Analytic Stability — Lyapunov's Direct Method

Definition 0.1 *The origin of the state space is stable if there exists a region, $S(r)$, such that states which start within $S(r)$ remain within $S(r)$.*

Definition 0.2 *Systems which satisfy Definition 0.1 are asymptotically stable if as $t \rightarrow \infty$, the system state approaches the origin of the state space.*

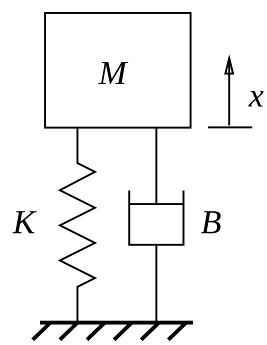
Analytic Stability

Definition 0.3 Lyapunov's Second Method *If the function, $V(\vec{x}, t)$, exists such that:*

$$
\begin{aligned}
V(\vec{0}, t) &= 0, \ and \\
V(\vec{x}, t) &> 0, \ for \ x \neq 0 \quad (positive \ definite), \ and \\
\partial V / \partial t &< 0 \quad (negative \ definite),
\end{aligned}
$$

then, the state space described by V is asymptotically stable in the neighborhood of the origin. If a system is stable, then there is a suitable Lyapunov function. If however, a particular Lyapunov function does not satisfy these criteria, it is not necessarily true that this system is unstable.

EXAMPLE: spring-mass-damper



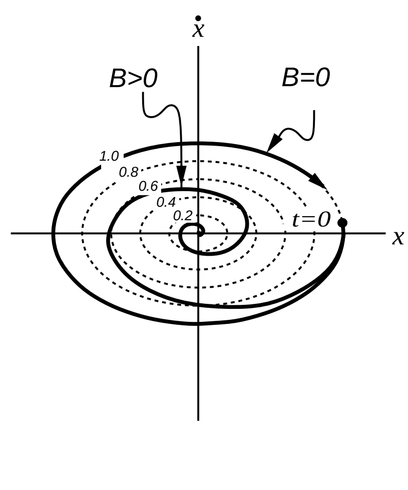
system dynamics:

$\ddot{x} + \frac{B}{M}\dot{x} + \frac{K}{M}x = 0$

quadratic Lyapunov function:

$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Kx^2$ *positive definite*

$$\begin{aligned} \frac{dL}{dt} &= M\dot{x}\ddot{x} + Kx\dot{x} = M\dot{x}(-\frac{B}{M}\dot{x} - \frac{K}{M}x) + Kx\dot{x} \\ &= -B\dot{x}^2 \qquad \qquad negative\ definite \end{aligned}$$



EXAMPLE: population dynamics

system dynamics:

$x_1 = \#$ males $\qquad \dot{x}_1 = -x_1 + \alpha x_1 x_2 = x_1(\alpha x_2 - 1)$

$x_2 = \#$ females $\qquad \dot{x}_2 = -x_2 + \beta x_1 x_2 = x_2(\beta x_1 - 1)$

equilibrium points: $\dot{\vec{x}} = \vec{0}$

(a) $x_1 = x_2 = 0$

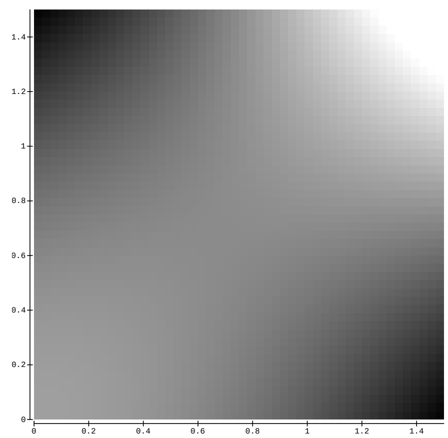
(b) $x_1 = \frac{1}{\beta}, x_2 = \frac{1}{\alpha}$

Lyapunov function:

$$\left.\begin{array}{l} V(\vec{0}, t) = 0 \\ V(\vec{x}, t) > 0 \end{array}\right\} \quad \text{choose } V(\vec{x}, t) = x_1^2 + x_2^2$$

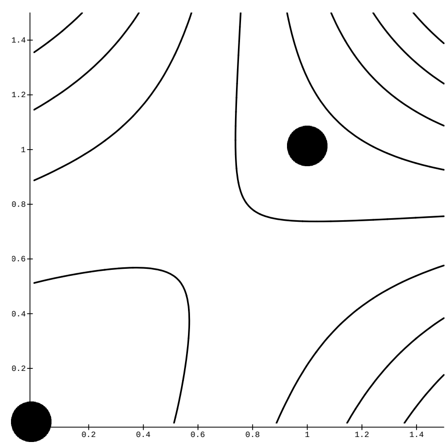
$$\begin{aligned} \frac{\partial V}{\partial t} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2x_1^2(\alpha x_2 - 1) + 2x_2^2(\beta x_1 - 1) \leq 0 \end{aligned}$$

EXAMPLE: population dynamics



level curves
in dV/dt function

x_2
\# females



x_1
\# males