Dynamics

The branch of physics that treats the action of force on bodies in motion or at rest; kinetics, kinematics, and statics, collectively. — Websters dictionary

Outline

- Conservation of Momentum
- Inertia Tensors
- Newton/Euler Dynamics
- State Space, Configuration Space, and Cartesian Forms
- Lagrangian Dynamics
- Computational issues

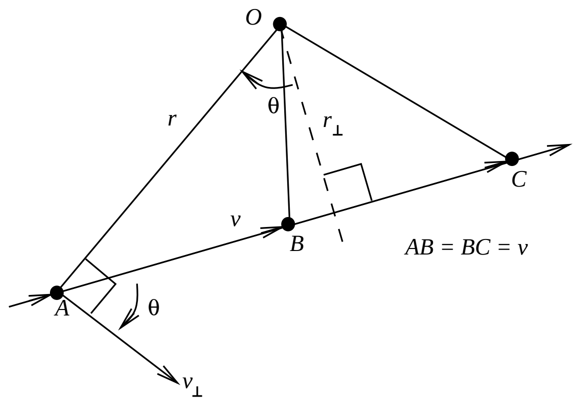
Newton's Laws

1. the particle will remain in a state of constant rectilinear motion unless acted on by an external force;

2. the time-rate-of-change in the momentum (mv) of the particle is proportional to the externally applied forces, $F = \frac{d}{dt}(mv)$;

3. and any force imposed on body A by body B is reciprocated by an equal and opposite reaction force on body B by body A.

Conservation of Angular Momentum

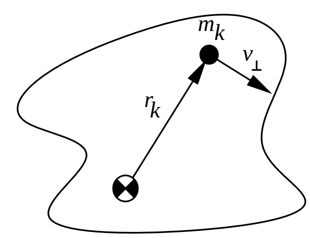


particle of mass m moving with uniform rectilinear velocity v ...

$$Area = \frac{1}{2} v r_\perp = \frac{1}{2} r v_\perp = constant$$

Therefore, the quantity, $\boxed{L = mrv_\perp}$ (angular momentum) is conserved.

For a collection of such particles:



$$L_{total} = \sum_k L_k = \sum_k m_k r_k v_{\perp,k} = \sum_k m_k r_k^2 \dot{\theta}_k$$

when $\dot{\theta}_1 = \dot{\theta}_2 = \ldots = \dot{\theta}_k$

$$L_{total} = \left(\sum_k m_k r_k^2 \right) \dot{\theta}$$

$$I = \sum_k m_k r_k^2$$

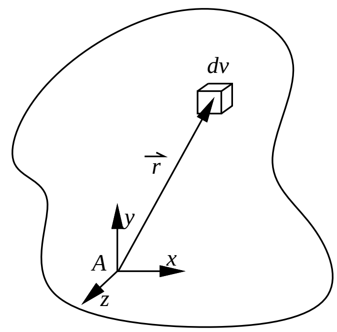
(rotational moment of inertia)

$[kg - m^2]$

and

$$\tau = \frac{d}{dt}\left[I\dot{\theta} \right] = I\ddot{\theta}$$

Inertia Tensor



MASS MOMENTS
OF INERTIA

$$I_{xx} = \int\int\int (y^2 + z^2)\rho dv$$

$$I_{yy} = \int\int\int (x^2 + z^2)\rho dv$$

$$I_{zz} = \int\int\int (x^2 + y^2)\rho dv$$

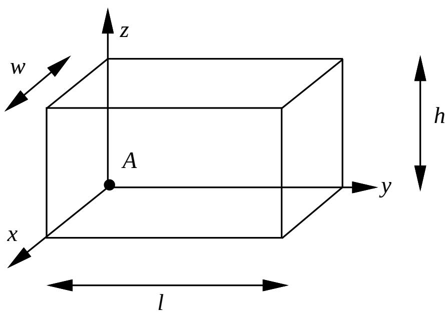
MASS PRODUCTS
OF INERTIA

$$I_{xy} = \int\int\int xy\rho dv$$

$$I_{xz} = \int\int\int xz\rho dv$$

$$I_{yz} = \int\int\int yz\rho dv$$

EXAMPLE:



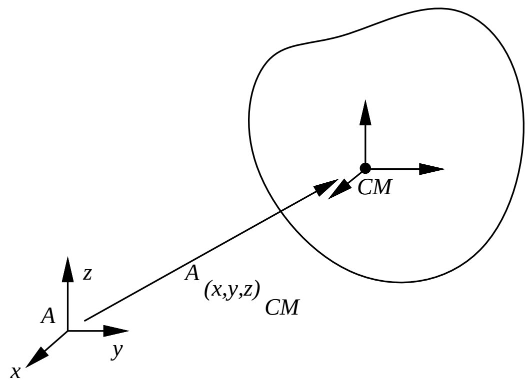
$$
\begin{aligned}
I_{xx} &= \int_0^h \int_0^l \int_0^w (y^2+z^2)\rho dx dy dz \\
&= \int_0^h \int_0^l (y^2+z^2) w\rho dy dz \\
&= \int_0^h \left[(\frac{y^3}{3} + z^2 y) \right]_0^l w\rho dz \\
&= \int_0^h (\frac{l^3}{3} + z^2 l) w\rho dz \\
&= \left(\frac{l^3 z}{3} + \frac{l z^3}{3} \right) \Big|_0^h (w\rho) \\
&= \left(\frac{l^3 h}{3} + \frac{l h^3}{3} \right) w\rho
\end{aligned}
$$

or, since the mass of the rectangle $m = (wlh)\rho$,

$$
I_{xx} = \frac{m}{3}\left(l^2 + h^2\right).
$$

$$
{}^A I = \begin{bmatrix} \frac{m}{3}(l^2+h^2) & \frac{m}{4}wl & \frac{m}{4}hw \\ \frac{m}{4}wl & \frac{m}{3}(w^2+h^2) & \frac{m}{4}hl \\ \frac{m}{4}hw & \frac{m}{4}hl & \frac{m}{3}(l^2+w^2) \end{bmatrix}
$$

Parallel Axis Theorem



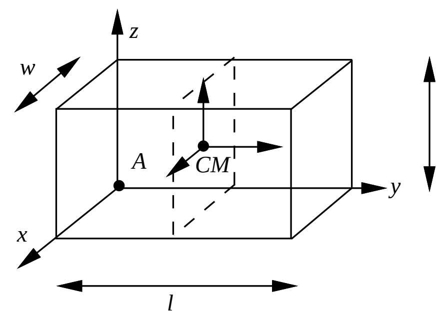
the moments of inertia look like

$$ {}^{A}I_{zz} = {}^{CM}I_{zz} + m({}^{A}x_{CM}^{2} + {}^{A}y_{CM}^{2}), $$

and the products of inertia are,

$$ {}^{A}I_{xy} = {}^{CM}I_{xy} + m({}^{A}x_{CM} \, {}^{A}y_{CM}). $$

EXAMPLE:



$$
\begin{aligned}
{}^{CM}I_{zz} &= {}^{A}I_{zz} - m({}^{A}x_{CM}^2 + {}^{A}y_{CM}^2) \\
&= \frac{m}{3}(l^2 + w^2) - \frac{m}{4}(l^2 + w^2) \\
&= \frac{m}{12}(l^2 + w^2)
\end{aligned}
$$

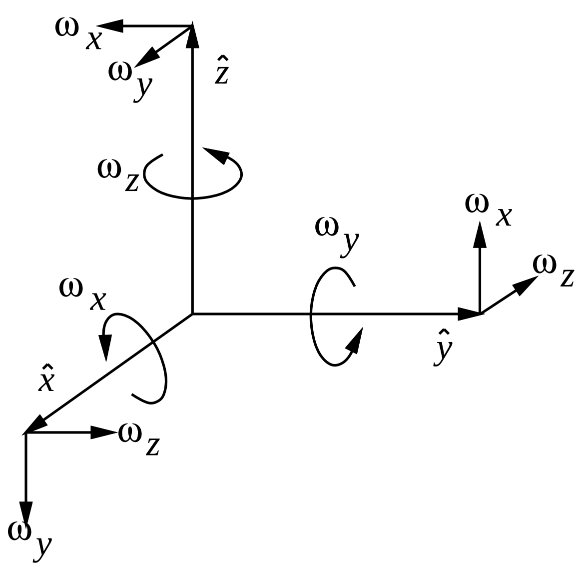
and

$$
\begin{aligned}
{}^{CM}I_{xy} &= {}^{A}I_{xy} - m({}^{A}x_{CM}\ {}^{A}y_{CM}) \\
&= \frac{m}{4}(wl) - \frac{m}{4}(wl) = 0.
\end{aligned}
$$

resulting in the diagonalized inertia tensor

$$
{}^{CM}I = \frac{m}{12}\begin{bmatrix} (l^2 + h^2) & 0 & 0 \\ 0 & (w^2 + h^2) & 0 \\ 0 & 0 & (l^2 + w^2) \end{bmatrix}
$$

Rotating Coordinate Systems



Consider a rotation about the $\hat{z}$ axis, an angular velocity ω_z displaces the $\hat{x}$ axis in the $\hat{y}$ direction

$$\dot{\hat{x}}_{\omega_z} = \omega_z \hat{y}$$

Similarly, ω_y displaces the $\hat{x}$ axis in the $-\hat{z}$ direction

$$\dot{\hat{x}}_{\omega_y} = -\omega_y \hat{z}$$

so that, in general,

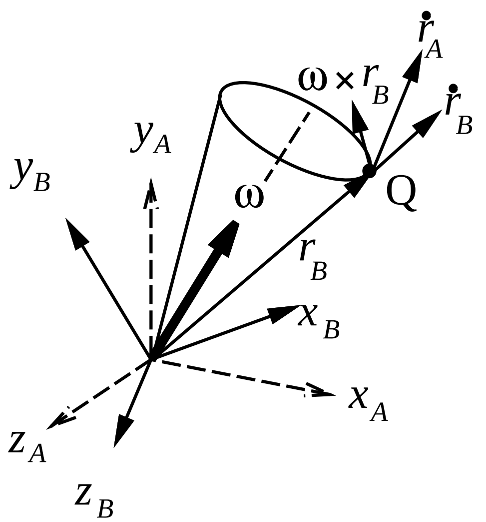
$$\begin{aligned}
\dot{\hat{x}} &= & \omega_z \hat{y} & -\omega_y \hat{z} \\
\dot{\hat{y}} &= -\omega_z \hat{x} & & \omega_x \hat{z} \\
\dot{\hat{z}} &= \omega_y \hat{x} & -\omega_x \hat{y} &
\end{aligned}$$

or, for arbitrary vectors $\vec{r} = r_x \hat{x} + r_y \hat{y} + r_z \hat{z}$ where r_x, r_y, r_z are constants in the coordinates of the rotating frame ...

$$\begin{aligned}
\dot{\vec{r}} &= \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \vec{r} \\
&= \vec{\omega} \times \vec{r}
\end{aligned}$$

Rotating Coordinate Systems: differentiation rule

Now, consider the case when vector $\vec{r_B}$ written in rotating frame B varies with respect to time:



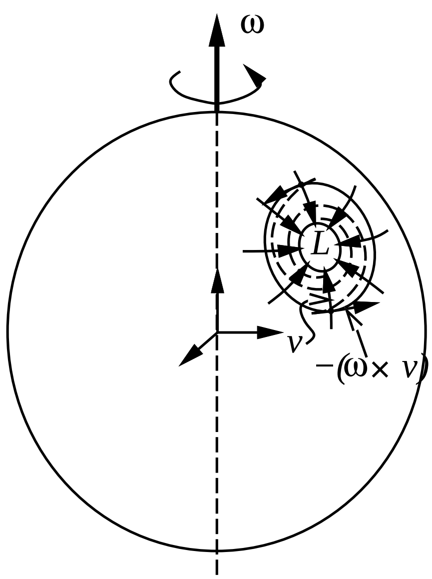
$$r_A =_A \mathbf{R}_B(t) r_B(t).$$

$$\begin{aligned}\dot{r}_A &= \frac{d}{dt}\left[{}_A\mathbf{R}_B(t) r_B(t)\right] \\ &= {}_A\mathbf{R}_B \dot{r}_B + \ {}_A\dot{\mathbf{R}}_B \ r_B \\ &= {}_A\mathbf{R}_B \left[\dot{r}_B + (\omega_B \times r_B)\right]\end{aligned}$$

$$\frac{d}{dt}\left[{}_A\mathbf{R}_B(t)(\cdot)_B\right] =_A \mathbf{R}_B \left[\frac{d}{dt}(\cdot)_B + (\omega_B \times (\cdot)_B)\right]$$

Rotating Coordinate Systems: EXAMPLE

Low pressure systems are regions in which large scale atmospheric flows converge. For the stationary (nonrotating) planet, this would result in flow lines directed radially inward.



But the earth rotates, so as each molecule in the atmosphere follows the pressure gradient, it will also experience a tangential acceleration. If we designate a stationary inertial frame A about which the earth frame (frame B) rotates;

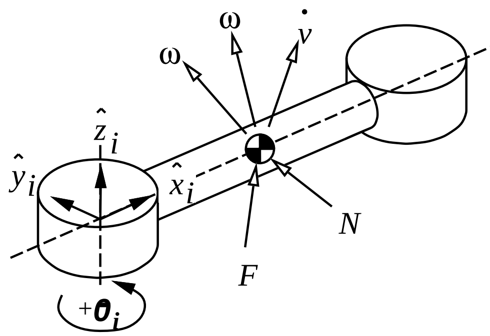
$$\begin{aligned}\vec{v}_A &= {}_A\mathbf{R}_B(t)\vec{v}_B \\ \dot{\vec{v}}_A &= {}_A\mathbf{R}_B[\dot{\vec{v}}_B + (\vec{\omega} \times \vec{v}_B)]\end{aligned}$$

so that an observer on the surface of the planet sees:

$$\dot{\vec{v}}_B = {}_B\mathbf{R}_A[\dot{\vec{v}}_A] - (\vec{\omega} \times \vec{v}_B)$$

The conjunction of a convergent flow and a rotating system, therefore, leads to a counterclockwise flow in the northern hemisphere. High pressure systems should rotate clockwise in the northern hemisphere, and just the opposite effects are observed in the southern hemisphere.

Newton/Euler Equations



Newton's Equation

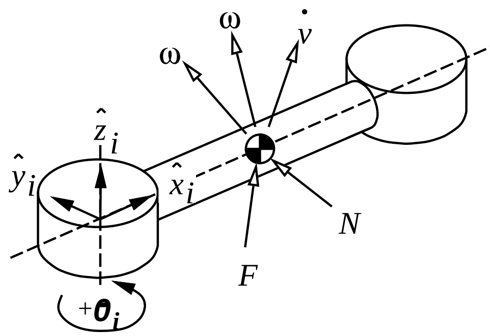
$$
\begin{aligned}
F &= \frac{d}{dt}\left(\mathbf{R}_i m_i v_i\right) = \mathbf{R}_i m_i \dot{v}_i + \dot{\mathbf{R}}_i m_i v_i \\
&= \mathbf{R}_i \left[m_i \dot{v}_i + (\omega_i \times m_i v_i) \right]
\end{aligned}
$$

Euler's Equation

$$
\begin{aligned}
N &= \frac{d}{dt}\left(\mathbf{R}_i M_i \omega_i\right) \\
&= \mathbf{R}_i \left[M_i \dot{\omega}_i + (\omega_i \times M_i \omega_i) \right]
\end{aligned}
$$

where F and N are the net force and torque, respectively acting upon link i written in inertial coordinates, and $\mathbf{R}_i$ is the rotation matrix relating frame i to the inertial frame, and ω_i is the total angular velocity of link i written in link i coordinates.

Newton/Euler Equations



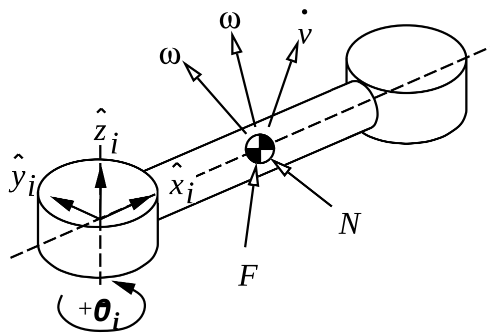
If F and N are written in the local coordinate frame for link i, then

$$\begin{bmatrix} mI & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} \omega \times mv \\ \omega \times M\omega \end{bmatrix} = \begin{bmatrix} F \\ N \end{bmatrix} = W$$

where $W \in R^6$ is the generalized force or *wrench* consisting of forces and torques acting on link i written in link i coordinates.

therefore, if we can account for the state of motion, $(\omega, \dot{\omega}, \dot{v})$, then we can compute the total load, W, acting on the center of mass and define the equation of motion for link i

Recursive Newton/Euler Equations



Assume that the absolute state of motion, $(\omega, \dot{\omega}, \dot{v})_i$, is known at frame i our goal is to write expressions for these quantities at frame $(i+1)$.

Angular Velocity: ω

$$REVOLUTE: \quad {}^{i+1}\omega_{i+1} = {}_{i+1}\mathbf{R}_i \, {}^{i}\omega_i + \dot{\theta}_{i+1}\hat{z}_{i+1}$$

$$PRISMATIC: \quad {}^{i+1}\omega_{i+1} = {}_{i+1}\mathbf{R}_i \, {}^{i}\omega_i$$

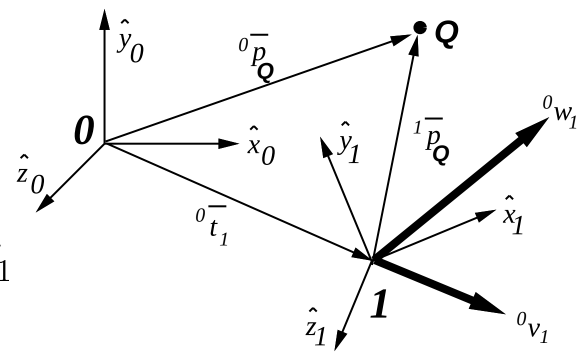
Angular Acceleration: $\dot{\omega}$

$$\begin{aligned} REVOLUTE: \quad {}^{i+1}\dot{\omega}_{i+1} &= {}_{i+1}\mathbf{R}_i \, {}^{i}\dot{\omega}_i + ({}_{i+1}\mathbf{R}_i \, {}^{i}\omega_i \times \dot{\theta}_{i+1}\hat{z}_{i+1}) \\ &+ \ddot{\theta}_{i+1}\hat{z}_{i+1} \end{aligned}$$

$$PRISMATIC: \quad {}^{i+1}\dot{\omega}_{i+1} = {}_{i+1}\mathbf{R}_i \, {}^{i}\dot{\omega}_i$$

Recursive Newton/Euler Equations: cont.

Linear Acceleration: $\dot{v}$



$$ {}^{0}\vec{p}_Q = {}_{0}\mathbf{R}_1 \, {}^{1}\vec{p}_Q + {}^{0}\vec{t}_1 $$

$$ {}^{0}\vec{v}_Q = {}_{0}\mathbf{R}_1 \, {}^{1}\dot{\vec{p}}_Q + ({}^{0}\omega_1 \times {}_{0}\mathbf{R}_1 \, {}^{1}\vec{p}_Q) + {}^{0}\vec{v}_1 $$

$$ \begin{aligned} {}^{0}\dot{\vec{v}}_Q &= \frac{d}{dt}\left[{}_{0}\mathbf{R}_1 \, {}^{1}\dot{\vec{p}}_Q \right] + ({}^{0}\dot{\omega}_1 \times {}_{0}\mathbf{R}_1 \, {}^{1}\vec{p}_Q) + ({}^{0}\omega_1 \times \frac{d}{dt}\left[{}_{0}\mathbf{R}_1 \, {}^{1}\vec{p}_Q \right]) \\ &\quad + {}^{0}\dot{\vec{v}}_1 \\ &= {}_{0}\mathbf{R}_1 \, {}^{1}\ddot{\vec{p}}_Q + ({}^{0}\omega_1 \times {}_{0}\mathbf{R}_1 \, {}^{1}\dot{\vec{p}}_Q) + ({}^{0}\dot{\omega}_1 \times {}_{0}\mathbf{R}_1 \, {}^{1}\vec{p}_Q) \\ &\quad + ({}^{0}\omega_1 \times {}_{0}\mathbf{R}_1 \, {}^{1}\dot{\vec{p}}_Q) + ({}^{0}\omega_1 \times {}^{0}\omega_1 \times {}_{0}\mathbf{R}_1 \, {}^{1}\vec{p}_Q) + {}^{0}\dot{\vec{v}}_1 \end{aligned} $$

Recursive Newton/Euler Equations: cont.

Linear Acceleration: $\dot{v}$

Now, substitute:

$$\begin{aligned} frame\ 0 &\Leftrightarrow frame\ (i-1) \\ frame\ 1 &\Leftrightarrow frame\ i \\ frame\ 2 &\Leftrightarrow frame\ (i+1) \end{aligned}$$

$$\begin{aligned} {}^{i+1}\dot{\vec{v}}_{i+1} &= {}_{i+1}R_{i-1}\Big[\, {}_{i-1}R_i\, {}^{i}\ddot{\vec{p}}_{i+1} + 2(\, {}^{i-1}\omega_i \times\ {}_{i-1}R_i\, {}^{i}\dot{\vec{p}}_{i+1}) \\ &\qquad +(\, {}^{i-1}\dot{\omega}_i \times\ {}_{i-1}R_i\, {}^{i}\vec{p}_{i+1}) \\ &\qquad +(\, {}^{i-1}\omega_i \times\ {}^{i-1}\omega_i \times\ {}_{i-1}R_i\, {}^{i}\vec{p}_{i+1}) +\ {}^{i-1}\dot{\vec{v}}_i\Big] \\ &= {}_{i+1}R_i\Big[\, {}^{i}\ddot{\vec{p}}_{i+1} + 2(\, {}^{i}\omega_i \times\ {}^{i}\dot{\vec{p}}_{i+1}) + (\, {}^{i}\dot{\omega}_i \times\ {}^{i}\vec{p}_{i+1}) \\ &\qquad +(\, {}^{i}\omega_i \times\ {}^{i}\omega_i \times\ {}^{i}\vec{p}_{i+1}) +\ {}^{i}\dot{\vec{v}}_i\Big] \end{aligned}$$

$$\begin{aligned} REVOLUTE: \quad & {}^{i}\vec{p}_{i+1} = const,\ {}^{i}\dot{\vec{p}}_{i+1} =\ {}^{i}\ddot{\vec{p}}_{i+1} = 0 \\ & {}^{i+1}\dot{\vec{v}}_{i+1} =\ {}_{i+1}R_i\Big[\, {}^{i}\dot{\vec{v}}_i + (\, {}^{i}\dot{\omega}_i \times\ {}^{i}\vec{p}_{i+1}) \\ &\qquad\qquad +(\, {}^{i}\omega_i \times\ {}^{i}\omega_i \times\ {}^{i}\vec{p}_{i+1})\Big] \end{aligned}$$

$$\begin{aligned} PRISMATIC: \quad & {}^{i}\vec{p}_{i+1} = d_i\hat{x}_i,\ {}^{i}\dot{\vec{p}}_{i+1} = \dot{d}_i\hat{x}_i,\ {}^{i}\ddot{\vec{p}}_{i+1} = \ddot{d}_i\hat{x}_i \\ & {}^{i+1}\dot{\vec{v}}_{i+1} =\ {}_{i+1}R_i\Big[\, {}^{i}\dot{\vec{v}}_i + \ddot{d}_i\hat{x}_i + 2(\, {}^{i}\omega_i \times \dot{d}_i\hat{x}_i) \\ &\qquad\qquad +(\, {}^{i}\dot{\omega}_i \times d_i\hat{x}_i) \\ &\qquad\qquad +(\, {}^{i}\omega_i \times\ {}^{i}\omega_i \times d_i\hat{x}_i)\Big] \end{aligned}$$

Recursive Newton/Euler Equations: cont.

Now, refer the translational acceleration to the center of mass:

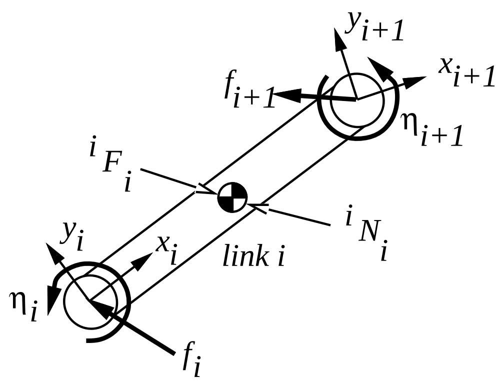
$$ {}^{i+1}\dot{v}_{cm,(i+1)} = ({}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}\vec{p}_{cm}) + ({}^{i+1}\omega_{i+1} \times {}^{i+1}\omega_{i+1} \times {}^{i+1}\vec{p}_{cm}) + {}^{i+1}\dot{v}_{i+1} $$

and we may write the Newton-Euler equation of motion:

$$ {}^{i+1}F_{i+1} = m_{i+1} \; {}^{i+1}\dot{v}_{cm\ i+1} $$

$$ {}^{i+1}N_{i+1} = M_{i+1} \; {}^{i+1}\dot{\omega}_{i+1} + ({}^{i+1}\omega_{i+1} \times M_{i+1} \; {}^{i+1}\omega_{i+1}) $$

Forces in Open Kinematic Chains



$$\sum Forces \;=\; {}^iF_i = {}^if_i - {}_iR_{i+1}\,{}^{i+1}f_{i+1},\ or$$
$$ {}^if_i = {}^iF_i + {}_iR_{i+1}\,{}^{i+1}f_{i+1}$$

$$\sum Torques = {}^iN_i = {}^i\eta_i - {}^i\eta_{i+1} - ({}^ip_{cm} \times {}^if_i) - (({}^ip_{i+1} - {}^ip_{cm}) \times {}^if_{i+1},$$

but, ${}^if_i = {}^iF_i + {}_iR_{i+1}\,{}^{i+1}f_{i+1}$, so that,

$$ {}^iN_i = {}^i\eta_i - {}^i\eta_{i+1} - ({}^ip_{cm} \times {}^iF_i) - ({}^ip_{i+1} \times {}^if_{i+1})$$

or,

$$ {}^i\eta_i = {}^iN_i + {}_iR_{i+1}\,{}^{i+1}\eta_{i+1} + ({}^ip_{cm} \times {}^iF_i) + ({}^ip_{i+1} \times {}_iR_{i+1}\,{}^{i+1}f_{i+1})$$

EXAMPLE: outward/inward iterations 2DOF, planar arm

Lagrangian Dynamics

Definition (Lagrangian) - The difference between the kinetic and potential energy of a dynamical system.

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

Theorem: (Lagrange's Equations) The equations of motion for a mechanical syste with generalized coordinates $q \in \mathcal{R}^m$ and Lagrangian, L are given by:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \Upsilon_i \qquad i = 1, \ldots, m$$

where Υ_i is the vector of external forces acting on the i^{th} generalized coordinate, g_i.

In vector coordinates,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{q}}} - \frac{\partial L}{\partial \vec{q}} = \vec{\Upsilon}$$

proof: Calculus of Variations (beyond the scope of this course)

Lagrangian Dynamics — some intuition

Suppose that our Lagrangian is the difference between some kinetic energy $\frac{1}{2}m\dot{q}^2$ and some potential energy mgq

$$\begin{aligned}
\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] &= \frac{\partial L}{\partial q} + \Upsilon \\
\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}m\dot{q}^2\right)\right] &= \frac{\partial}{\partial q}(mgq) + \Upsilon \\
\frac{d}{dt}(m\dot{q}) &= mg + \Upsilon \\
\frac{d}{dt}(momentum) &= applied\ force \qquad \textbf{Newton's Equation}
\end{aligned}$$

EXAMPLE: Lagrangian dynamics
2 DOF planar arm