Chapter 6

Control Theory

In this chapter, we examine tools for control synthesis in the frequency domain. The 1 Degree-of-Freedom (DOF) direct-drive robot manipulator is used to illustrate control synthesis, specifically the issues surrounding the composition of a discrete time controller and a continuous plant.

6.1 Closed Loop Transfer Functions

Figure 6.1 is a schematic representing the basic structure of all feedback control systems.

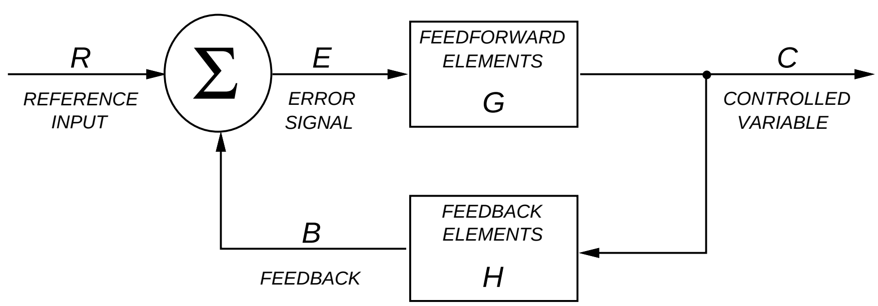


Figure 6.1 *Generalized Feedback Control Structure*

The input reference state, R, is compared to the feedback state, generating an error signal, E. This error is supplied to the feedforward elements of the controller, G, which generally consists of some control stage composed with the physical device which is to be regulated. The result is a state of the system from which we derive the controlled variable, C. The feedback elements of the controller transform the controlled variable into a form consistent with the reference input. Elements G and H are often referred to as the forward and feedback transfer functions, respectively, since they transform input signals into another form via differential equations expressing the dynamics of the system. If we are given an input reference signal, then by inspection of Figure 6.1:

$$C = G\,E \tag{6.1}$$

$$B = H\,C, \textit{ and} \tag{6.2}$$

$$E = R - B \tag{6.3}$$

and the *closed loop transfer function* (cltf) can be derived:

$$\frac{C}{R} = \frac{GE}{R} = \frac{G(R-B)}{R} = \frac{G}{R/(R-B)} = \frac{G}{(E+B)/E} = \frac{G}{1+B/E} = \frac{G}{1+HC/E}, \textit{ or} \tag{6.4}$$

$$\frac{C}{R} = \frac{G}{1+GH} \tag{6.5}$$

This result suggests that the general feedback control structure may always be expressed as the equivalent feedforward system depicted in Figure 6.2. This form of the transfer function relates the controlled variable, C, directly to the reference input, R.

$$\frac{C}{R} = \frac{G}{1+GH} \quad \Rightarrow \quad (1+GH)C = (G)R \tag{6.6}$$

If we consider the "unforced" differential equation of motion for the regulated system, that is, $R = 0$, we find the relationship $(1+GH)C = 0$ which is simply the homogeneous form of the equation of motion. This implies that the characteristic equation for this system is expressed in the denominator of the closed loop transfer function, $(1+GH)$ and, as we shall see, the stability and the response of the closed loop system are determined completely by this characteristic equation.

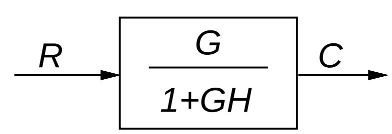


Figure 6.2 *The Closed Loop Transfer Function*

6.2 The 1 DOF Direct-Drive Robot in the Time Domain

In the sections to follow, we will consider the implications of computational latency on the stability of feedback systems. The model for our discussion will be a simplified model of a 1 DOF, direct drive robot arm. This system which serves as the basis for this model is illustrated in Figure 6.3.

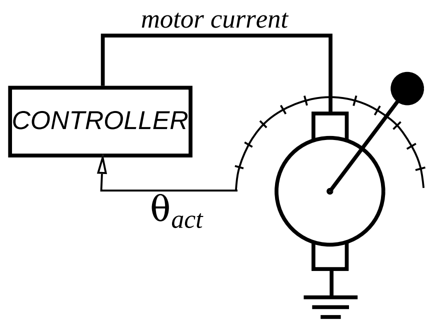


Figure 6.3 *The 1 DOF Direct-Drive Robot Arm*

This idealized motor produces a torque, τ, proportional to the error between the reference and actual motor positions.

$$\tau = A(\theta_{ref} - \theta_{act}) = Ae, \tag{6.7}$$

where A is the amplifier gain. This torque produces an angular acceleration of the motor armature, J, against a viscous (damper) load, B.

$$\tau = J\ddot{\theta}_{act} + B\dot{\theta}_{act} \tag{6.8}$$

The feedback control loop which expresses this system model is pictured in Figure 6.4.

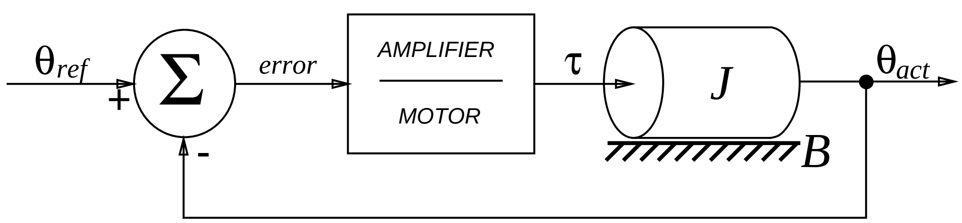


Figure 6.4 *A Feedback Controller for the 1 DOF Direct-Drive Robot Arm*

Combining Equations 6.7 and 6.8, and using the differential operator, D, yields

$$(JD^2 + BD)\theta_{act}(t) = \tau = Ae. \tag{6.9}$$

The feedforward transfer function in Figure 6.4 transforms the error, e, into θ_{act}. Using Equation 6.9 we may write the feedforward transfer function as follows:

$$\begin{aligned} G &= \frac{\theta_{act}}{e} = \frac{A}{JD^2 + BD} = \frac{A/J}{D(D + B/J)} \\ &= \frac{K}{D(D+a)}, \end{aligned} \tag{6.10}$$

where $K = A/J$ and $a = B/J$. The feedback transfer function, H, is just unity in this case, so we may write the closed loop transfer function as:

$$\begin{aligned} \frac{\theta_{cont}}{\theta_{ref}} &= \frac{G}{1+GH} \\ &= \frac{\frac{K}{D(D+a)}}{1+\frac{K}{D(D+a)}} \\ &= \frac{K}{D(D+a)+K} \end{aligned} \tag{6.11}$$

6.3 Laplace Transform

If $f(t)$ is a function of time, t, then the Laplace transform, $F = \mathcal{L}[f(t)]$, is defined by:

$$F(s) = \int_0^\infty f(t)e^{-st}dt. \tag{6.12}$$

The variable, s, is a complex quantity of the form, $\sigma + j\omega$. The Laplace integral will converge if $f(t)$ is piecewise continuous, and is of exponential order — i.e., there exists an a such that $e^{-at}|f(t)|$ is bounded for all $t < T$ where T is some finite time. A linear differential equation with constant coefficients and a finite number of terms is Laplace-transformable.

Table 6.1 includes many functions which are commonly encountered in linear system analysis. it is

Table 6.1: Laplace Transform Pairs

Name	f(t)	F(s)
unit impulse	$\delta(t)$	1
unit step	$u(t)$	$\frac{1}{s}$
ramp	t	$\frac{1}{s^2}$
n^{th}-order ramp	t^n	$\frac{n!}{s^{n+1}}$
exponential	e^{-at}	$\frac{1}{s+a}$
ramped exponential	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$
sine	$\sin at$	$\frac{a}{s^2+a^2}$
cosine	$\cos at$	$\frac{s}{s^2+a^2}$
damped sine	$e^{-at}sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
damped cosine	$e^{-at}sin\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
hyperbolic sine	$\sinh at$	$\frac{a}{s^2-a^2}$
hyperbolic cosine	$\cosh at$	$\frac{s}{s^2-a^2}$

generally possible to solve for a function, $f(t)$, whose Laplace transform is a given rational function of s, by converting it into a sum of partial fractions and then finding functions, $f(t)_i$, whose Laplace transform is each partial fraction.

Table 6.2 summarizes three important theorems related to the Laplace transform which will come in handy in subsequent examples.

Table 6.2: Laplace Transform Theorems

Name	Theorem
Derivative	$\mathcal{L}[\frac{df}{dt} = sF(s) - f(0+)$
Integral	$\mathcal{L}\left[\int_0^t f(t)dt\right] = s^{-1}F(s)$
Shifting	$\mathcal{L}[f(t-t_0)u(t-t_0)] = e^{-t_0 s}F(s)$

6.4 The 1 DOF Direct-Drive Robot in the Frequency Domain

In Table 6.2 we see

$$\mathcal{L}[Df(t)] = s\mathcal{L}[f(t)]$$

This identity allows us to transform the transfer equation for the 1 DOF, direct-drive robot,

$$\frac{\theta_{act}}{\theta_{ref}} = \frac{K}{D(D+a)+K}. \tag{6.13}$$

The result is:

$$\frac{\Theta(s)_{act}}{\Theta(s)_{ref}} = \frac{K}{s(s+a)+K}. \tag{6.14}$$

Figure 6.5 illustrates the control loop transfer function expressed in the frequency domain.

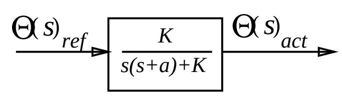


Figure 6.5 *The closed-loop transfer function in the frequency domain.*

EXAMPLE: *Suppose that at $t = 0$, we apply a unit step reference input to our direct drive robot. The transform of the unit step is:*

$$u(t) = 1$$
$$U(s) = \frac{1}{s}$$

Therefore, if we let $K = (A/J) = 1$ and $a = (B/J) = 2$ in Equation 6.14

$$\Theta(s)_{cont} = \frac{1}{s^2(s+2)+1} = \frac{1}{s(s+1)^2} \tag{6.15}$$

A partial-fraction expansion of this quotient yields:

$$\begin{aligned}\Theta(s)_{cont} &= \frac{1}{s(s+1)^2} = \frac{a}{s} + \frac{b}{(s+1)} + \frac{c}{(s+1)^2} \\ &= \frac{1}{s} + \frac{-1}{(s+1)} + \frac{-1}{(s+1)^2}\end{aligned} \tag{6.16}$$

The inverse Laplace transform of these terms yields (from the tables):

$$\theta(t)_{cont} = 1 - e^{-t} - te^{-t} \tag{6.17}$$

so that at $t = 0$, $\theta(t)_{cont} = 0$, *but as* $t \to \infty$, *the direct drive robot converges to the reference input.*

Recall that if we consider the "unforced" differential equation of motion for the regulated system, that is, $R = 0$, we find the relationship $(1 + GH)C = 0$ which is simply the homogeneous form of the equation of motion. This implies that the characteristic equation for this system is expressed in the denominator of the closed loop transfer function, $(1 + GH)$ and that the stability and the response of the closed loop system are determined completely by this characteristic equation. In the previous example,

$$\frac{G}{1+GH} = \frac{\Theta(s)_{cont}}{\Theta(s)_{ref}} = \frac{K}{s(s+a)+K}. \tag{6.18}$$

The characteristic equation of the system is then:

$$s(s+a) + K = s^2 + as + K = 0. \tag{6.19}$$

In the example above, with $K = 1$ and $a = 2$, the characteristic equation was $s^2 + 2s + 1 = (s+1)^2$. The repeated root of this characteristic equation, $s = -1$, is what gives rise to the negative exponential terms in the time domain solution. In fact, these roots must be negative, since positive roots will yield positive exponentials which suggests that the time domain solution will be unbounded as $t \to \infty$.

It should be clear that the nature of the system response and the stability of the system is directly related to the roots of the characteristic equation. These roots are a function of $K = (A/J)$ in our direct drive robot (the ratio of the amplifier gain to the robot's rotational inertia).
If we let $a = (B/J) = 2$, and allow K to vary between $-\infty$ and $+\infty$ the Laplace transform yields the *root locus* pictured in Figure 6.6.
The system stability is directly dependent on the selection of K. Moreover, for each stable controller there is a range of performance characteristics (speed, overshoot, compliance, settling time, etc.) which also depend on the K employed.

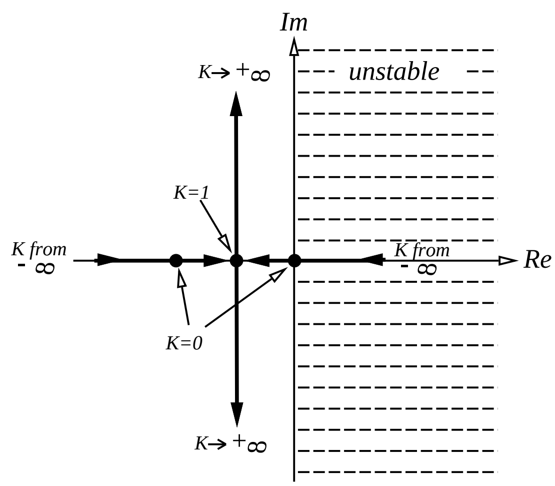


Figure 6.6 *The Root Locus for the 1 DOF, Direct Drive Robot Arm*

6.5 Digital Controllers

Figure 6.7 illustrates a digital implementation of the control loop we have been developing for the 1 DOF direct-drive robot arm. The diagram represents the integration of a digital control system and a continuous plant. A continuous motor position error is regularly sampled and converted into a digital signal. The control algorithm transforms the digital position error into a digital command which is subsequently converted back into a continuous signal and applied to the plant. While a digital controller is typically much easier to implement and to modify, there are costs associated with this convenience. There are two related, fundamental design issues associated with digital controllers.

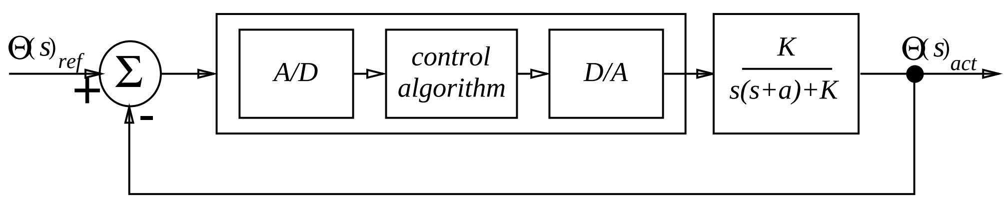


Figure 6.7 *A Digital Controller for the 1 DOF, Direct Drive Robot Arm*

1. The controller only looks at the state of the system and the reference input signal at discrete instants. This implies that there are limitations to the frequency of input and state fluctuations observable by this system. The *Sampling Theorem* requires that the controller sampling frequency be at least twice the highest frequency component in the reference input or in the plant response. If events in the world occur too quickly for a fixed sampling rate, then these events are not completely observable and may induce instabilities in the system.

2. Aside from these observability problems, it takes a finite amount of time following an observation before a control response can be generated. The plant will no longer be in the state that elicited this particular control action. This effect can be modeled as a pure delay in the signal propagation. We will illustrate this effect with the following example.

6.5.1 Computational Latency

EXAMPLE: *The Laplace transform of a signal that is delayed by a fixed time is derived from the following:*

$$F(s) = \int_0^\infty f(t - t_0)e^{-st}dt.$$

If we make a change of variables, $t' = t - t_0$, we may write this as:

$$
\begin{aligned}
F'(s) &= \int_{t_0}^{\infty} f(t')e^{-s(t'+t_0)}dt \\
&= e^{-st_0}\int_{t_0}^{\infty} f(t')e^{-st'}dt' \\
&= e^{-st_0}\int_{0}^{\infty} f(t')e^{-st'}dt' + C \\
&= e^{-st_0}F(s) + C
\end{aligned} \tag{6.20}
$$

A feedback control loop with an infinite sampling rate and a pure computational latency is illustrated in Figure 6.8. The open-loop transfer function for this model is then:

$$G = \frac{Ke^{-st_0}}{s(s+a)},$$

and the closed-loop transfer function is:

$$
\begin{aligned}
\frac{G}{1+G} &= \frac{Ke^{-st_0}/(s(s+a))}{1+Ke^{-st_0}/(s(s+a))} \\
&= \frac{Ke^{-st_0}}{s^2+as+Ke^{-st_0}}
\end{aligned} \tag{6.21}
$$

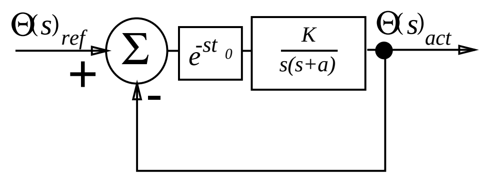


***Figure 6.8** The Feedback Control Loop with a Computational Delay*

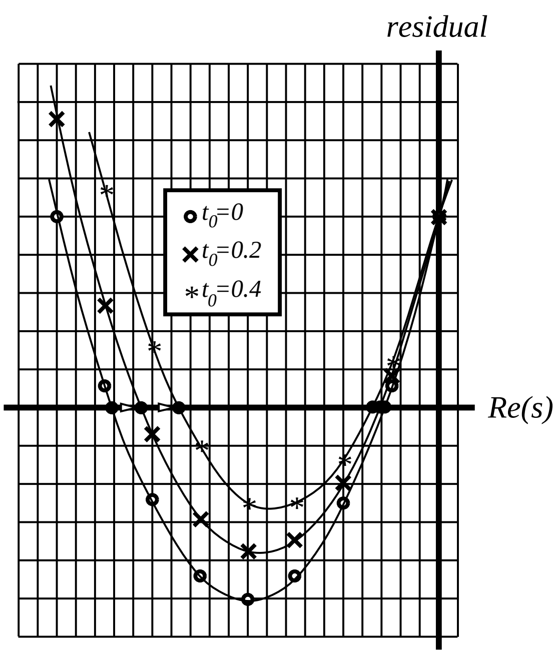


***Figure 6.9** The Characteristic Equation as a Function of Computational Delay*

Once again, the denominator of Equation 6.21 is the characteristic equation for the controlled system.

$$\underbrace{s^2+as+K}_{original}\underbrace{e^{-st_0}}_{delay} \tag{6.22}$$

When the delay period, $t_0 = 0$, this characteristic equation is identical to the original (Equation 6.19). But for latencies greater than zero, the exponential term modifies the roots of the characteristic equation. A complete analysis of the effect of the delay would require tools that we haven't developed in these notes. However, we can appreciate the destabilizing effect of the delay by noticing what happens to the roots of the characteristic equation. The original system (see Figure 6.6) was stable for $a = 2$ and $K = 0.5$, where its characteristic equation produced two negative, real roots. Figure 6.9 is a plot of the real part of Equation 6.22 as a function of s for various delay times, t_0.

This example illustrates that the most stable root is displaced in the positive direction as computational latency increases. Since each real root will produce an exponential term in the time domain (an earlier example illustrated this), the delay destabilizes the system at a fixed gain. Moreover, the qualitative response of the system will also change as result, eventually producing an oscillation. If the controller is scheduled on a distributed architecture for which computational latency is not predictable, then it follows that the response of the system will not be predictable either.

The Effects of Sampling — Z-transform

D'Azzo and Houpis "Feedback Control System Analysis and Synthesis," McGraw-Hill, Inc, 1966, p. 687-695
a very concise demonstration of sampling and the z-transform analysis for the 1 DOF direct-drive robot example.

6.6 Actuators

6.6.1 Permanent Magnet DC Motors

A particularly popular actuator for robot systems is based on the Lorentz force produced when a current loop is placed in a magnetic field. Figure 6.10 illustrates the electromechanical configuration of the DC motor. Electrical energy is transformed into mechanical energy by pushing an electrical charge through the loop (or armature) in the presence of a directed magnetic field. The Lorentz force, $qV \times B$, generates a torque on the rotor proportional to the current (qV) and the magnetic field strength.

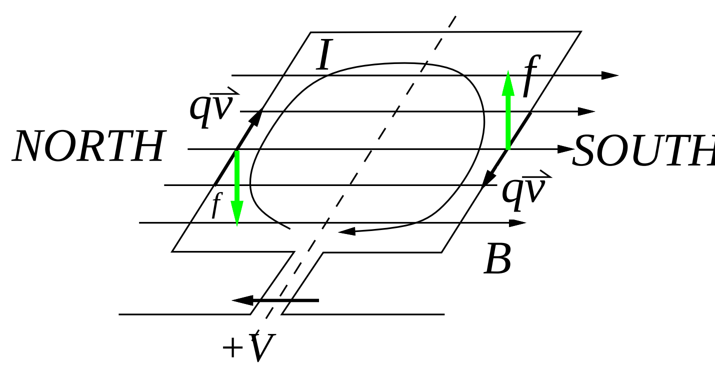


Figure 6.10 *A schematic of a single loop in the rotor.*

The opposite is true, if a force is input to the system, such as is done in a wind turbine, for instance, when mechanical energy causes the rotor to revolve and the resulting velocity through the magnetic field causes electrons to accelerate (i.e., current). In fact, this property affects the DC motor as well since this velocity induced current is in the opposite direction as the current moving in the conductor. This effect can be modeled as a voltage that is proportional to the angular velocity of the rotor — referred to as the *back emf*.

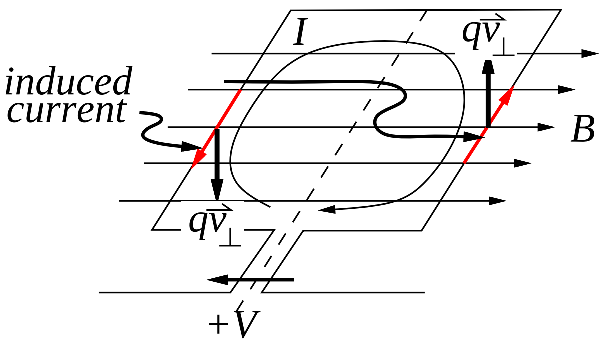


Figure 6.11 *Backwards electromotive force due to rotational velocity.*

Figure 6.11 shows an induced *current* perpendicular to the axis of the conductor that is the result of the rotor velocity. A simple application of the right hand rule verifies that the current thus produced is opposite in direction to the current that generated the motion in the first place. This suggests that the ability of a motor to generate torque will diminish as the motor velocity increases.

This very simple model also suggests that we may expect the steady state torque produced in the motor to be propotional to the amount of current we can push through the loop, and that its steady state velocity will be proportional to the voltage. In practice, this motor runs out of torque as soon as the current loop becomes perpendicular to the magnetic field. At this point, or slightly before, a reversal of the current in the coil can continue to accelerate the rotor in the same direction. This is referred to as commutation and can be accomplished in solid state switching circuits or mechanically using brushes and conducting pads.

The electrodynamics of the DC motor are derived by approximating the motor as a lumped parameter system. The relationship between torque and current in the motor is a (very linear) function of the number of windings, the magnetic field strength and the current supplied by the motor driver, $\tau = K_t I$, where τ is the rotor torque, K_t is the torque constant for the motor, and I is the current.

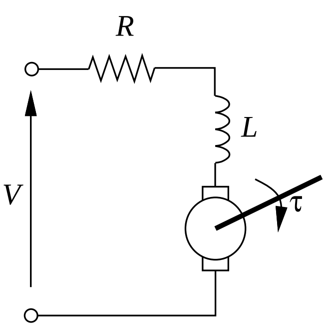


Figure 6.12 *Electrical model of the DC motor.*

The back emf (or "generator" effect mentioned earlier) is also, as it turns out, a linear function of the rotor velocity, $V_b = K_b\dot{\theta}$. The induced current flow induces a backwards voltage potential, V_b, proportional to the angular velocity $\dot{\theta}$ through the constant, K_b. Writing the sum of voltages around the circuit yields:

$$V = IR + L\frac{dI}{dt} + K_b\dot{\theta} \tag{6.23}$$

Often, the inductance introduced by the motor windings is negligible and can be omitted.

Under these circumstances, the loop equation becomes:

$$V \approx IR + K_b\dot{\theta} \tag{6.24}$$

By relating the the mechanical power out $\tau\dot{\theta}$ to the electrical power input VI and resistive losses, we can derive the relationship between K_t and K_b.

$$\begin{array}{ccccc} \textit{mechanical} & & \textit{electrical} & & \\ \textit{power} & = & \textit{power} & - & \textit{losses} \\ \textit{out} & & \textit{in} & & \\ & & & & \\ \tau\dot{\theta} & = & VI & - & I^2R \\ (K_tI)\dot{\theta} & = & (IR + K_b\dot{\theta})I & - & I^2R \\ & = & K_bI\dot{\theta} & & \\ K_t & = & K_b & & \end{array}$$

The dynamic equation of motion for the motor can be derived by summing all the torque applied to the rotor and employing Equation 6.24:

$$\begin{aligned} \sum \tau = J\ddot{\theta} &= KI \\ &= K\left[\frac{V}{R} - \frac{K\dot{\theta}}{R}\right] \end{aligned}$$

Rearranging terms, we get:

$$\ddot{\theta} + \frac{K^2}{JR}\dot{\theta} + \frac{KV}{JR} = 0 \tag{6.25}$$

The rotational moment of inertia, J, includes the rotor inertia, the gearbox load, and the external load. Which of these dominates the system's dynamics is an interesting question.

Typical DC motor applications drive the load through a reduction or gearbox. This implies that the motor dynamics must consider what a compound load such as that depicted in Figure 6.13 looks like to the motor when viewed through the gearbox.

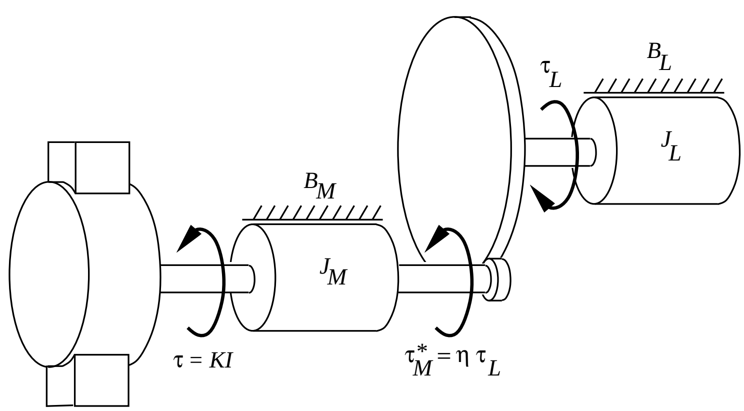


Figure 6.13 *The compound load of the motor-gearhead combination*

If we consider the transmission to be perfectly efficient and linear, then the motor applies the torque produced by the Lorentz forces to accelerate the rotor and load inertias and to overcome the viscous friction of the rotor and load as well. The load is driven through a gearbox reduction $\eta < 1$. The "reduction" is in rotational velocity since

$$\begin{aligned} \theta_L &= \eta\theta_M, \\ \dot{\theta}_L &= \eta\dot{\theta}_M, \; and \\ \ddot{\theta}_L &= \eta\ddot{\theta}_M \end{aligned}$$

However, if the transmission is perfectly efficient, then the power input is equal to the power output, or

$$\begin{aligned} \tau_{out}\omega_{out} &= \tau_{in}\omega_{in} \\ \tau_{out}(\eta\omega_{in}) &= \tau_{in}\omega_{in} \\ \tau_{out} &= \tau_{in}\frac{\omega_{in}}{(\eta\omega_{in})} \\ \tau_{out} &= \tau_{in}\frac{1}{\eta} \end{aligned} \tag{6.26}$$

Therefore, the transmission amplifies the output torque; if the reduction $\eta = 0.01$, then the output shaft carries one hundred times the torque at one hundredth the velocity of the input shaft. We may write the dynamic equation of motion for this compound load by equating the torque derived from Lorentz forces on the rotor with the torques consumed to accelerate the load and dissipated in the viscous friction.

$$\tau = \left[J_M\ddot{\theta}_M + B_M\dot{\theta}_M\right] + \eta\left[J_L\ddot{\theta}_L + B_L\dot{\theta}_L\right] \tag{6.27}$$

Where the second term is the external load, τ_L, referred to the motor shaft, $\tau_M^* = \eta\tau_L$, as is shown in the diagram. By using the velocity relationship across the transmission ($\theta_L = \eta\theta_M$), and rearranging terms:

$$\begin{aligned} \tau &= \left[J_M\ddot{\theta}_M + B_M\dot{\theta}_M\right] + \eta^2\left[J_L\ddot{\theta}_M + B_L\dot{\theta}_M\right] \\ &= \left[J_M + \eta^2 J_L\right]\ddot{\theta}_M + \left[B_M + \eta^2 B_L\right]\dot{\theta}_M \end{aligned}$$

where:

$$\begin{aligned} J_{eff} &= J_M + \eta^2 J_L \\ B_{eff} &= B_M + \eta^2 B_L \end{aligned}$$

For large reductions (η small) the inertia of the compound load is dominated by the rotor. This is very significant, since the rotor inertia is not dependent on the robot configuration, unlike the load inertia.

We may also wish to determine how much torque is required to *backdrive* the system: how much torque is required on the load shaft to accelerate the compound load. This is precisely the same analysis, except that we will reference torque and velocity to the load shaft rather than the motor shaft.

$$\begin{aligned}\tau_{in} &= \left[J_L\ddot{\theta}_L + B_L\dot{\theta}_L\right] + \frac{1}{\eta}\left[J_M\ddot{\theta}_M + B_M\dot{\theta}_M\right] \\ &= \left[J_L\ddot{\theta}_L + B_L\dot{\theta}_L\right] + \frac{1}{\eta^2}\left[J_M\ddot{\theta}_L + B_M\dot{\theta}_L\right] \\ &= \left[J_L + \frac{1}{\eta^2}J_M\right]\ddot{\theta}_L + \left[B_L + \frac{1}{\eta^2}B_M\right]\dot{\theta}_L\end{aligned}$$

Therefore, from the perspective of the output shaft, a 100:1 reduction ($\eta = 0.01$) effective amplifies the rotor inertia $10,000$ times. This implies that actuators with large reductions are passively stiff since the rotor behaves like a massive flywheel. In these situations, compliance to external perturbations can only be accomplished through feedback compensators.

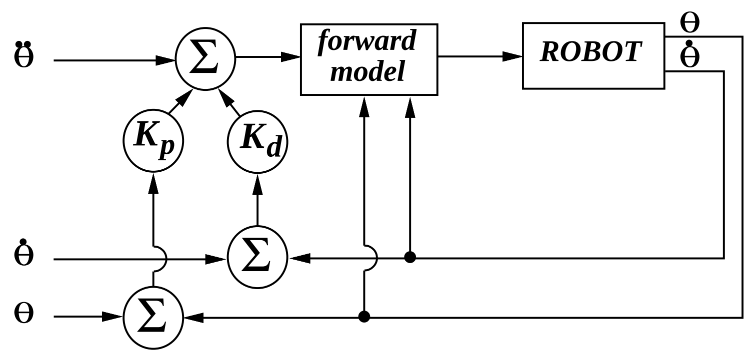
6.7 Homework Exercises

1. **Linear Feedback Controllers for Roger**
   Roger-the-Crab's dynamics were determined in Chapter **??** and in a previous programming assignment, you coded a feedforward compensator which both linearized and de-coupled this highly nonlinear dynamical system. Now we will begin to put Roger's eyes and arms to work. This programming assignment involves eye and arm kinematics, the implementation of a feedback controller, and the empirical verification of the resulting second-order, closed-loop response.

   This project requires a brief report that should discuss the follwing sequence of tasks.

   (a) Write a control procedure (a stub is provided) to implement the PD control portion of the feedback controller shown schematically in Figure 6.14 for both the eyes and the arms.

   $$\tau = K_p(\theta_{ref} - \theta_{act}) - K_d\dot{\theta}_{act}$$



(b) **Visual Attention**

Begin by designing a model-based PD controller for the eyes. Roger can see "objects" in the world that you place by clicking the left mouse button (see the README). The eyes have a focal length of 64 pixels and an image plane 128 pixels wide. So a vector defining the heading to your "object" within the field of view can be found. Two such headings can locate the (x, y) coordinate of the object.

Write a PD controller that *foveates* Roger's eyes on any single object that you place in the field of view by computing an angular error from the middle of the image plane. Select gains (springs and dampers) that behave nicely. Once both eyes have foveated, compute the resulting object (x, y) coordinate.

(c) **Visually Guided Reaching**

Given the object's (x, y) coordinate, decide which arm is appropriate for this goal. Through the inverse kinematic relation for this arm, compute a reference joint angle configuration and run that arm, via a PD controller to the perceived object. Again, select gains for the arm that work well no matter where the goal is placed. If everything works out, you should get a tactile response when the arm converges to the object.

(d) **Theoretical Verification**

Using the compensated (linearized) PD control, you may now elicit under- over- and critically-damped responses for the arm system by varying the gains, K_p and K_d, as described in the text. Every degree of freedom in the arm should match the theory for a linearized and de-coupled system.

(e) **Integrated Reach to a Visual Target**

On a single *error* vs *time* plot, show the transient response of the 6 DOF system from the time you place an object until you observe a tactile response. Error is just $\theta_{ref} - \theta_{act}$ for every degree of freedom in the system. Report the gains which produce these responses.