Chapter 5

Stability

We will look at Lyapunov's methods for expressing the stability of dynamical systems in the time domain. A prototypical dynamic system, the spring-mass-damper will be introduced to provide an example of this form of analysis. We will then examine tools for control synthesis in the frequency domain. The 1 Degree-of-Freedom (DOF) direct-drive robot manipulator is used to illustrate control synthesis, specifically the issues surrounding the composition of a discrete time controller and a continuous plant.

5.1 Lyapunov's Direct Method

Simply stated, Lyapunov's perspective is captured in the following two observations which comprise his "Direct Method."

Definition 5.1 *The origin of the state space is stable if there exists a region, $S(r)$, such that states which start within $S(r)$ remain within $S(r)$.*

Definition 5.2 *Systems which satisfy Definition 5.1 are asymptotically stable if as $t \to \infty$, the systems state approaches the origin of the state space.*

Definition 5.1 establishes a criterion for stability which requires that the state of the system never leaves a region of bounded size. This definition prohibits the system from diverging toward infinity. Stated another way, this condition requires that the total energy in the system remains bounded. Definition 5.2 is a stronger condition. An asymptotically stable system's state will approach the origin of the state space over time. This is equivalent to requiring that the system energy must decay over time to zero at an equilibrium state. It is not strictly necessary that the region of the state space in which the system operates contain the origin since a change of variables (translation of the origin) can always be defined which causes this region to envelop the origin. Likewise, it is not strictly required for the equilibrium state to have zero energy, only that it be the local minimum in the region of the state space under consideration.

5.2 Lyapunov's Second Method

The definition employs the *Lyapunov function* which is defined as an arbitrary scalar field written in terms of the state variables, $V(\vec{x}, t)$, that is continuous in all first derivatives. Notice that energy is an acceptable Lyapunov function by this definition.

Definition 5.3 ***Lyapunov's Second Method*** *Iff the function,* $V(\vec{x}, t)$*, exists such that:*

$$\begin{aligned} V(\vec{0}, t) &= 0, \; \textit{and} \\ V(\vec{x}, t) &> 0, \; \textit{for } x \neq 0 \quad (\textit{positive definite}), \; \textit{and} \\ \partial V / \partial t &< 0 \quad (\textit{negative definite}), \end{aligned}$$

then, the state space described by V *is asymptotically stable in the neighborhood of the origin. If a system is stable, then there is a proper Lyapunov function. If however, a particular Lyapunov function does not satisfy these criteria, it is not necessarily true that this system is unstable.*

In order to understand the utility of these observations we will introduce the most commonly analyzed dynamical system, the spring-mass-damper, and determine whether or not it constitutes a stable system.

5.3 The Spring-Mass-Damper System

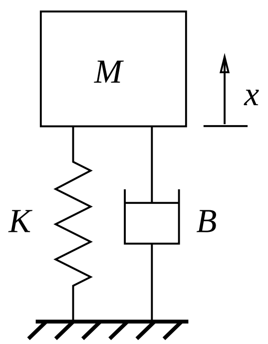

Figure 5.1
Spring-Mass-Damper System

Figure 5.1 illustrates a dynamical system consisting of a mass, M, a spring, K, and a damper, B. This is an idealization of the mechanism used in the suspension of your automobile or the mechanism used to close your screendoor without slamming. The equation of motion is derived by cataloging the forces that are produced by each of the three components of this system. The mass expresses the relationship between the force and acceleration acting on it,

$$F_m = Ma = M\ddot{x}.$$

The damper relates force to the velocity of deformation,

$$F_b = -Bv = -B\dot{x},$$

and the spring relates force to deformation,

$$F_k = -Kx.$$

To derive the equation of motion, all we must do is enumerate all of the force acting on mass, M. If the mass depicted in Figure 5.1 is accelerated in the $+\hat{x}$ direction, it will accumulate a velocity and a displacement in the $+\hat{x}$ direction as well. This hypothetical displacement will elongate the spring and damper in the system model. If we treat the mass as a free body and identify all the force impressed upon it, we end up with the system illustrated in Figure 5.2.

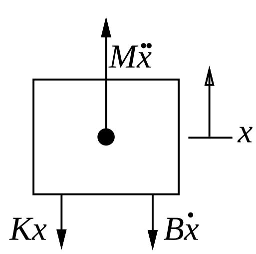

Figure 5.2 *Free Body Diagram of the Spring-Mass-Damper*

The net force on the mass is responsible for the upward acceleration in Figure 5.2 and since this is a free body, this force must be exactly equal to the sum of the forces generated by the spring and the damper.

$$\sum F = M\ddot{x} = -B\dot{x} - Kx$$

If we rearrange terms, this equality defines the homogeneous (or unforced) equation of motion.

$$M\ddot{x} + B\dot{x} + Kx = 0, \; or$$

$$\ddot{x} + (B/M)\dot{x} + (K/M)x = 0 \tag{5.1}$$

This second order differential can be written in another way:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \; where: \tag{5.2}$$

$$\zeta = B/2(KM)^{1/2}, \textit{ damping coefficient, and}$$
$$\omega_n = (K/M)^{1/2}, \textit{ the natural (resonant) frequency.}$$

Lyapunov's result suggests a means of analyzing the system's stability without explicitly solving the equation of motion. The energy, E, of this can be expressed as the sum of a kinetic energy and a potential energy:

$$E = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Kx^2 \tag{5.3}$$

The first term is kinetic energy (the integral of momentum over a change of velocity), while the second term represents the potential energy stored in the elastic deformation of the spring (computed as the integral of a force over a displacement). Note that we have expressed energy in Equation 5.3 in terms of the state variables $(x, \dot{x})$. Moreover, this expression describes an ellipse in the state space whose area is defined by the value of E (see Figure 5.3). If we differentiate Equation 5.3 with respect to time,

$$\frac{dE}{dt} = M\dot{x}\ddot{x} + Kx\dot{x},$$

and insert the expression for $\ddot{x}$ derived from Equation 5.1, we find:

$$\begin{aligned}\frac{dE}{dt} &= M\dot{x}(-(B/M)\dot{x} - (K/M)x) + Kx\dot{x} \\ &= -B\dot{x}^2 \end{aligned} \tag{5.4}$$

The rate of change in the energy of this system is negative definite since both B and $\dot{x}^2$ are positive real numbers.

To evaluate this result from the perspective of Lyapunov's Definitions 5.1 and 5.2, consider Figure 5.3. Since the derivative of energy with respect to time is negative definite, if we release this system at time 0 with energy 1.0, for instance, we may guarantee that it will never leave the envelope corresponding to an energy of 1.0. Moreover, if $B = 0$ then the system's state would orbit along the $E = 1.0$ ellipsoid continuously trading strain energy in the spring for kinetic energy. If however $B > 0$, then the system's state space trajectory would spiral spiral toward the origin and come to rest with zero energy — satisfying Lyapunov's criteria for asymptotic stability expressed in Definition 5.2. Therefore, the linear spring-mass-damper system is asymptotically stable over the entire state space.
This is not necessarily the case for all spring-mass-damper systems.

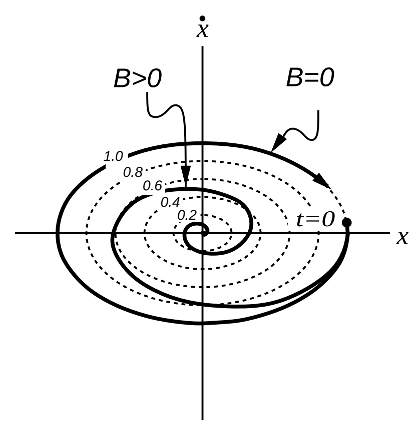

Figure 5.3 *The State Space Trajectory of the Spring-Mass-Damper System*

Consider non-linear viscoelastic components defined by:

$$F_k = Kx^3 \tag{5.5}$$

$$F_b = \alpha(1 - x^2)\dot{x}, \tag{5.6}$$

this damper is a function of the x component of the state space, $B(x) = \alpha(1 - x^2)$, $\alpha > 0$. Immediately, note that if $x > 1$ the system exhibits a negative B! Therefore, the damper will effectively inject energy into the system — violating both Lyapunov criteria for stability.

Formally, the equation of motion for this system becomes

$$M\ddot{x} + Kx^3 + \alpha(1 - x^2)\dot{x} = 0, \tag{5.7}$$

and the energy of the system is expressed as

$$\begin{aligned} E &= \int_0^v Mv'dv' + \int_0^x F_k dx', \text{ or} \\ &= \int_0^{\dot{x}} M\dot{x}'d\dot{x}' + \int_0^x Kx'^3 dx' \end{aligned}$$

so that for this system we find:

$$E = \frac{1}{2}M\dot{x}^2 + \frac{1}{4}Kx^4 \tag{5.8}$$

When we differentiate the energy with respect to time, we obtain:

$$\begin{aligned} \frac{dE}{dt} &= M\dot{x}\ddot{x} + Kx^3\dot{x} \\ &= M\dot{x}[-(K/M)x^3 - (\alpha/M)(1 - x^2)\dot{x}] + Kx^3\dot{x} \\ &= -Kx^3\dot{x} - \alpha(1 - x^2)\dot{x}^2 + Kx^3\dot{x} \\ \frac{dE}{dt} &= -\alpha(1 - x^2)\dot{x}^2 \end{aligned} \tag{5.9}$$

Figure 5.4 illustrates the stability regimes that emerge for this system as a function of position in the state space.

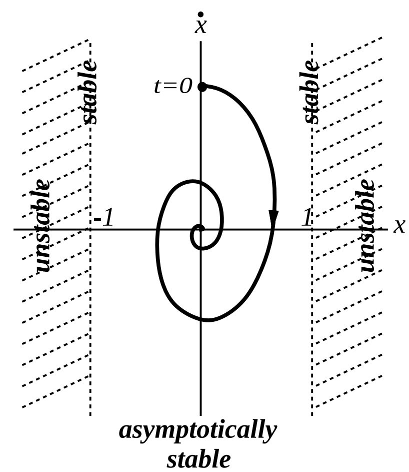

Figure 5.4 *A State Space Trajectory of the Nonlinear Spring-Mass-Damper System*