Chapter 4

Dynamical Systems

Definition 4.1 (Dynamics) *the branch of physics that treats the action of force on bodies in motion or at rest; kinetics, kinematics, and statics, collectively. — Websters dictionary*

On the basis of Chapter **??**, we are now in a position to analyze the affect of mass and force on kinematically complex bodies. The basis of this analysis is the work of Sir Isaac Newton whose principles of particle motion establish the relationship between force and acceleration. A generalization of these ideas in the form of Newton-Euler equations permits the construction of the dynamic equations of motion for articulated structures.

4.1 Newton's Laws

Consider a body idealized as a point mass moving in R^3. This *particle* undergoes pure translational motion. According to Newton's laws:

1. the particle will remain in a state of constant rectilinear motion unless acted on by an external force;
2. the time-rate-of-change in the momentum (mv) of the particle is proportional to the externally applied forces, $F = \frac{d}{dt}(mv)$;
3. and any force imposed on body A by body B is reciprocated by an equal and opposite reaction force on body B by body A.

In systems for which mass is constant, the second law provides the basis for one of the most significant equations in physics, $F = ma$, often referred to as Newton's equation. At first glance, the first law appears to be a restatement of the second law. However, the first law asserts an important prerequisite to the application of the second law. It states that a particle that experiences no external forces will move in a constant velocity *along a straight line.* If an observer of such a particle undergoes an acceleration and/or a rotation, then it the particle will appear to trace a *curved* path. The first law, therefore, establishes a property of the **inertial** or absolute coordinate frame. This property must be established in order for the second law to be applicable.

Newton's third law requires that when two particle interact, each experiences reciprocal forces equal in magnitude and opposite in direction. This property will be important when we consider articulated mechanical structures that transmit forces from one link to the next.

4.2 Euler's Equation

Consider rigid bodies that consist of a distribution of mass in space. Under translation, these bodies behave precisely the same as a particle of the same total mass located at the bodies center of mass. However, under rotation, the distribution of mass about the center of rotation is important. In this situation, Newton's equation is insufficient. To correctly describe the relationship between force and acceleration in rotating bodies, we must develop the notion of angular momentum. Figure 4.1 illustrates a particle of mass m moving with uniform rectilinear velocity v.

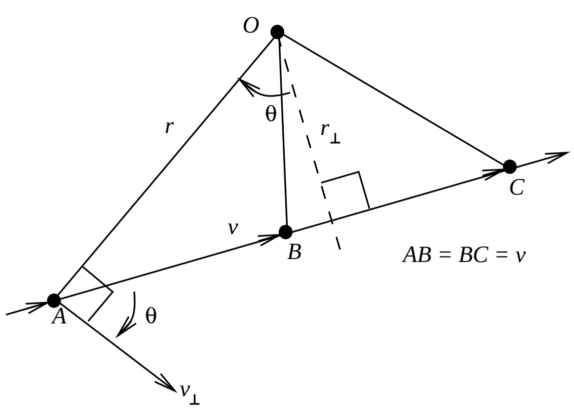


Figure 4.1 *Conservation of Angular Momentum.*

The area of any triangle defined by two successive positions of the particle and vertex O is a constant,

$$A = \frac{1}{2} v r_\perp = \frac{1}{2} r v_\perp.$$

It is clear that the quantity, $rv_\perp$ (or $mrv_\perp$) is conserved under this state of motion. In fact, it can also be shown to be conserved when the particle is under the influence of a central force directed toward O.The quantity, $mrv_\perp$ is reminiscent of linear momentum because it contains mv, but it is different. Momentum is in $[kg - m/sec]$, whereas $mrv_\perp$ is in $[kg - m^2/sec]$.

This quantity is the *angular momentum* of the particle, and must be defined with respect to a particular point (O in this case).

$$L = mrv_\perp$$

Now, consider a planar lamina consisting of a planar collection of point masses related through rigid body constraints as shown in Figure 4.2. The total angular momentum of this system is the linear sum of the angular momenta of the constituent point masses.

$$L_{total} = \sum_k L_k = \sum_k m_k r_k v_{\perp,k}$$

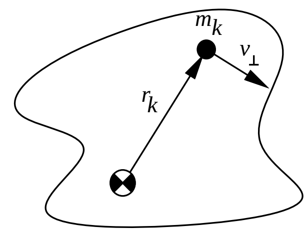


Figure 4.2 *A Collection of Point Masses in a Planar Lamina.*

Since, $\dot{x} = r\dot{\theta}$, we may express $v_{\perp,k} = r_k\dot{\theta}_k$ so that,

$$L_{total} = \sum_k m_k r_k^2 \dot{\theta}_k$$

For rigid bodies, $\dot{\theta}_1 = \dot{\theta}_2 = \ldots = \dot{\theta}_k$, so

$$L_{total} = \left(\sum_k m_k r_k^2\right)\dot{\theta} = M\dot{\theta}. \tag{4.1}$$

The quantity $M = \sum_k m_k r_k^2$ in units of $[kg - m^2]$ is the *rotational moment of inertia*. It is independent of the state of motion — an intrinsic property of the body, and is dependent on the distribution of mass about point O. For systems with fixed mass, we define the torque applied to the system to be the time rate of change in angular momentum.

$$\tau = \frac{d}{dt}\left[M\dot{\theta}\right] = M\ddot{\theta} \tag{4.2}$$

Equation 4.2 (Euler's equation) is the equivalent of Newton's equation, $F = m\ddot{x}$ for the rotating lamina. It states that the body remains in a constant state of rotation unless acted upon by a torque, which causes a corresponding angular acceleration.

4.3 Rotational Moment of Inertia

When we extend the analysis of the lamina into three dimensions, the body is now free to rotate about any axis in R^3. Once a *center of rotation* is identified, the moment of inertia of the body is dependent on the distribution of mass about that point. The effective moment of inertia varies as the axis of rotation varies. The inertia tensor permits us to write an expression for the moment of inertia about arbitrary axes[1].

[1] Mathematically, a tensor is used to describe the directional anisotropy in a property, in this case, the ability to transform torques into angular accelerations.

Figure 4.3 illustrates a 3D object that is rotating about the origin of frame A.

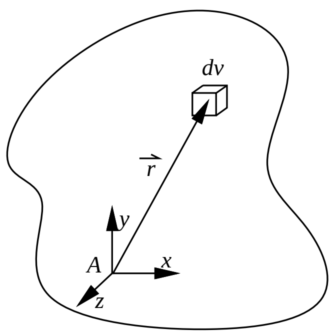


Figure 4.3 *A differential volume rotating about the origin of frame A.*

The differential volume element, dv, shown in the figure is assume to have a uniform mass density, ρ, so that the differential mass $dm = \rho dv$. This continuous mass distribution requires that we write the rotational moment of inertia, $M = \sum_k m_k r_k^2$, as an integral over the object volume.

MASS MOMENTS OF INERTIA	MASS PRODUCTS OF INERTIA
$M_{xx} = \int\int\int (y^2+z^2)\rho dv$	$M_{xy} = \int\int\int xy\rho dv$
$M_{yy} = \int\int\int (x^2+z^2)\rho dv$	$M_{xz} = \int\int\int xz\rho dv$
$M_{zz} = \int\int\int (x^2+y^2)\rho dv$	$M_{yz} = \int\int\int yz\rho dv$

EXAMPLE: The rectangular block illustrated in Figure 4.4 rotates about the origin of frame A. All the mass of the rectangle is in the positive octant of the coordinate system. For this case, we may write:

$$\begin{aligned}
M_{xx} &= \int_0^h \int_0^l \int_0^w (y^2+z^2)\rho dx dy dz \\
&= \int_0^h \int_0^l (y^2+z^2) w\rho dy dz \\
&= \int_0^h \left[(\frac{y^3}{3} + z^2 y) \right]_0^l w\rho dz \\
&= \int_0^h (\frac{l^3}{3} + z^2 l) w\rho dz \\
&= \left(\frac{l^3 z}{3} + \frac{l z^3}{3} \right) |_0^h (w\rho) \\
&= \left(\frac{l^3 h}{3} + \frac{l h^3}{3} \right) w\rho
\end{aligned}$$

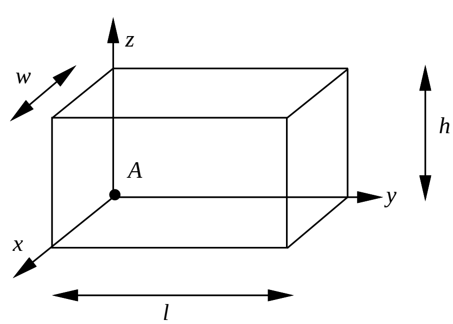


Figure 4.4 *A rectangular mass and a center of rotation A.*

The other elements of the inertia tensor can be computed in a similar fashion. Since the mass of the rectangular body is $m = (lwh)\rho$, we find that:

$$^{A}M = \begin{bmatrix} \frac{m}{3}(l^2+h^2) & \frac{m}{4}wl & \frac{m}{4}hw \\ \frac{m}{4}wl & \frac{m}{3}(w^2+h^2) & \frac{m}{4}hl \\ \frac{m}{4}hw & \frac{m}{4}hl & \frac{m}{3}(l^2+w^2) \end{bmatrix}$$

4.3.1 The Parallel Axis Theorem

The inertia tensor describes the moment of inertia about an arbitrary axis passing through the origin of frame A in Figure 4.4. However, if we wish to move the center of rotation, and therefore change the effective distribution of mass, we do not have to recompute the tensor from scratch.

In fact, the inertia tensor is tabulated in handbooks for commonly occuring shapes with respect to the body center of mass. The quantity is easily modified to reflect arbitrary rotation centers by noting that the body behaves like a lumped, point mass concentrated at the body's center of mass. Using the nomenclature of Figure 4.5, the moments of inertia can be written

$$^{A}M_{zz} = {}^{CM}M_{zz} + m({}^{A}x_{CM}^2 + {}^{A}y_{CM}^2),$$

and the products of inertia are,

$$^{A}M_{xy} = {}^{CM}M_{xy} + m({}^{A}x_{CM}\,{}^{A}y_{CM}).$$

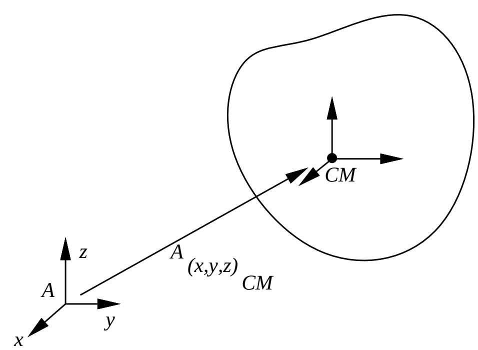


Figure 4.5 *An eccentric mass distribution.*

EXAMPLE: Suppose that the center of rotation for the rectangle is now moved to the object's center of mass. In this situation, the parallel axis theorem suggests that

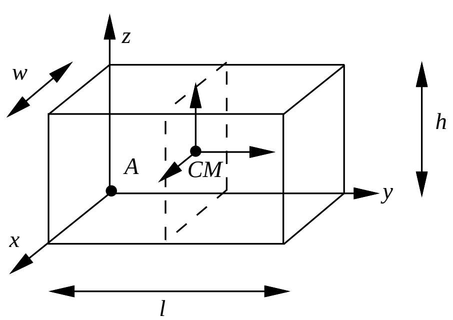


Figure 4.6 *Moving the center of rotation to the center of mass.*

$$\begin{aligned} ^{CM}M_{zz} &= {}^{A}M_{zz} - m({}^{A}x_{CM}^2 + {}^{A}y_{CM}^2) \\ &= \frac{m}{3}(l^2+w^2) - \frac{m}{4}(l^2+w^2) \\ &= \frac{m}{12}(l^2+w^2) \end{aligned}$$

and

$$\begin{aligned} ^{CM}M_{xy} &= {}^{A}M_{xy} - m({}^{A}x_{CM}\,{}^{A}y_{CM}) \\ &= \frac{m}{4}(wl) - \frac{m}{4}(wl) = 0. \end{aligned}$$

The result of moving the center of rotation to the center of mass is the diagonalized inertia tensor

$$^{CM}M = \frac{m}{12}\begin{bmatrix} (l^2+h^2) & 0 & 0 \\ 0 & (w^2+h^2) & 0 \\ 0 & 0 & (l^2+w^2) \end{bmatrix}$$

4.4 Rotating Coordinate Systems

If we assume that coordinate frame B in Figure 4.7 is moving with a constant angular velocity ω with respect to an inertial frame A, then the velocity of particle Q in inertial coordinates ($\dot{r}_A$) consists of its translational velocity in frame B ($\dot{r}_B$) plus a term due to the angular velocity of frame B ($\omega \times r_B$). This second term is due directly to the angular velocity of frame B — an observer attached to frame B cannot correctly account for the forces on particle Q by observing its apparent motion.

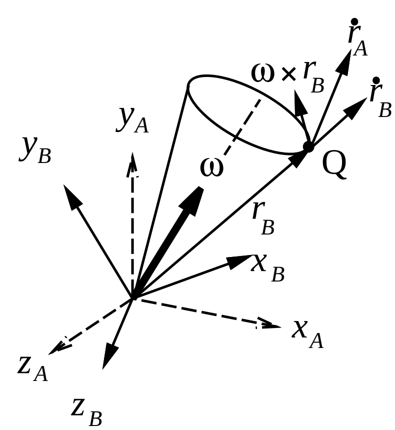


Figure 4.7 *Velocity in an inertial frame due to a time varying rotation matrix.*

From another perspective, we may consider coordinate frames A and B to be related by way of a time varying rotation matrix

$$r_A = {}_A\mathbf{R}_B(t) r_B.$$

The velocity of Q with respect to frame A is then:

$$\begin{aligned} \dot{r}_A &= \frac{d}{dt}\left[{}_A\mathbf{R}_B(t) r_B\right] \\ &= {}_A\mathbf{R}_B \dot{r}_B + {}_A\dot{\mathbf{R}}_B\, r_B \\ &= {}_A\mathbf{R}_B\left[\dot{r}_B + (\omega_B \times r_B)\right] \end{aligned}$$

In the final expression for $\dot{r}_A$, we have expressed the absolute angular velocity of frame B in frame B coordinates for notational convenience.

Again we find that the velocity of particle Q relative to the inertial frame consists of the translational velocity of Q within the B coordinate system plus a term inherited from frame B's state of motion. This result holds for any vector quantity that is expressed in a local coordinate system which is itself rotating with respect to an inertial frame of reference.

$$\frac{d}{dt}\left[{}_A\mathbf{R}_B(t)(\cdot)_B\right] = {}_A\mathbf{R}_B\left[\frac{d}{dt}(\cdot)_B + (\omega_B \times (\cdot)_B)\right]$$

4.5 Newton-Euler Equations of Motion

Figure 4.8 depicts a single link (link i) of an articulated structure and identifies the parameters necessary to completely describe its instantaneous state of motion. The local coordinate frame for link i is attached to its proximal end, with it's $\hat{z}$ axis aligned with a revolute degree of freedom. The dynamic parameters are referred to the center of mass of the link.

The net force and the time rate of change in velocity are related through Newton's equation:

$$F = \frac{d}{dt}(mv) = m\dot{v} \tag{4.3}$$

which states that the force acting on this free body is equivalent to the time rate of change in the body's linear momentum. In rotating coordinate frames, however, we must also account for the velocity due to the angular velocity of frame i. In particular, if frame i in Figure 4.8 has angular velocity $\mathbf{R}_i\omega_i$ with respect to the inertial frame, then

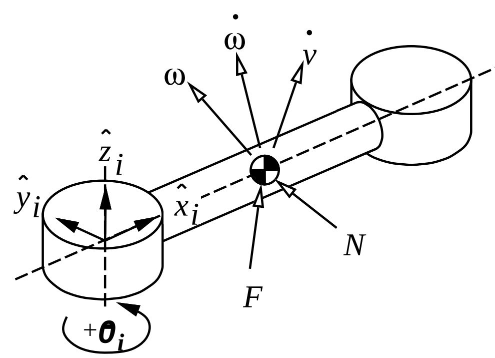


Figure 4.8 *Free body diagram of link i.*

$$\begin{aligned} F &= \frac{d}{dt}(\mathbf{R}_i m_i v_i) = \mathbf{R}_i m_i \dot{v}_i + \dot{\mathbf{R}}_i m_i v_i \\ &= \mathbf{R}_i \left[m_i \dot{v}_i + (\omega_i \times m_i v_i) \right] \end{aligned} \tag{4.4}$$

is Newton's law for body i, where F is the net force acting upon link i written in inertial coordinates, $\mathbf{R}_i$ is the rotation matrix relating frame i to the inertial frame, and ω_i is the absolute angular velocity of link i written in link i coordinates.

Euler's law is the analog of Newton's equation, relating angular acceleration to torque through the rotational moment of inertia.

$$\begin{aligned} N &= \frac{d}{dt}(\mathbf{R}_i M_i \omega_i) \\ &= \mathbf{R}_i \left[M_i \dot{\omega}_i + (\omega_i \times M_i \omega_i) \right] \end{aligned} \tag{4.5}$$

is Euler's equation for body i, where N is the net torque acting upon link i written in inertial coordinates, and $\mathbf{R}_i$ is the rotation matrix relating frame i to the inertial frame.

Equations 4.4 and 4.5 can be combined to yield the equations of motion of the link. If F and N are written in the local coordinate frame for link i, then

$$\begin{bmatrix} mI & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} + \begin{bmatrix} \omega \times mv \\ \omega \times M\omega \end{bmatrix} = \begin{bmatrix} F \\ N \end{bmatrix} = W \tag{4.6}$$

where $W \in R^6$ is the generalized force or *wrench* consisting of forces and torques acting on link i written in link i coordinates.

4.5.1 Propagating Velocities in Open Kinematic Chains

If we define the complete state of motion in a mechanism, that is all positions, velocities, and accelerations, then we can compute the forces in the structure resulting from this state of motion. The Newton-Euler equations are the basis for a recursion in which $(\omega, \dot{\omega}, \dot{v})_i$ can be computed for $i = 0, n$ using the state of motion in the immediately proximal link, $(\omega, \dot{\omega}, \dot{v})_{i-1}$. This recursion begins at the inertial frame where $(\omega_0 = 0, \dot{\omega}_0 = 0, \dot{v} = -\vec{g})$. The inertial frame is given an acceleration equal and opposite to gravity. This is mathematically equivalent to operating the robot in a uniform gravitational field but allows us to avoid accounting for individual gravity loads on each link.

In the following derivations, we will assume that the absolute state of motion, $(\omega, \dot{\omega}, \dot{v})_i$, is known at frame i and our goal is to write expressions for these quantities at frame $(i + 1)$.

Angular Velocity: ω

$$REVOLUTE: \quad {}^{i+1}\omega_{i+1} = {}_{i+1}\mathbf{R}_i \, {}^{i}\omega_i + \dot{\theta}_{i+1}\hat{z}_{i+1}$$

$$PRISMATIC: \quad {}^{i+1}\omega_{i+1} = {}_{i+1}\mathbf{R}_i \, {}^{i}\omega_i$$

Angular Acceleration: $\dot{\omega}$

$$\begin{aligned} REVOLUTE: \quad {}^{i+1}\dot{\omega}_{i+1} &= {}_{i+1}\mathbf{R}_i \, {}^{i}\dot{\omega}_i + (-\dot{\theta}_{i+1}\hat{z}_{i+1} \times {}_{i+1}\mathbf{R}_i \, {}^{i}\omega_i) + \ddot{\theta}_{i+1}\hat{z}_{i+1} \\ &= {}_{i+1}\mathbf{R}_i \, {}^{i}\dot{\omega}_i + ({}_{i+1}\mathbf{R}_i \, {}^{i}\omega_i \times \dot{\theta}_{i+1}\hat{z}_{i+1}) + \ddot{\theta}_{i+1}\hat{z}_{i+1} \end{aligned}$$

$$PRISMATIC: \quad {}^{i+1}\dot{\omega}_{i+1} = {}_{i+1}\mathbf{R}_i \, {}^{i}\dot{\omega}_i$$

Linear Acceleration: $\dot{v}$ Consider the motion of particle Q moving with respect to a coordinate frame, 1, which is in turn moving with respect to an *inertial* frame, 0, as depicted in Figure 4.9.

Here ${}^0\omega_1$ and ${}^0v_1$ are the angular the translational velocity respectively, of frame 1 expressed in frame 0 coordinates.
Given ${}^1\vec{p}_Q$, the position vector for point Q in frame 1 coordinates, then

$$ {}^0\vec{p}_Q = {}_0\mathbf{R}_1\ {}^1\vec{p}_Q + {}^0\vec{t}_1 $$

where ${}^0\vec{t}_1$ is just the position vector of frame 1 written in frame 0 coordinates. For general velocities in R^6, the velocity of particle Q expressed in frame 0 can be written:

$$ {}^0\vec{v}_Q = {}_0\mathbf{R}_1\ {}^1\dot{\vec{p}}_Q + ({}^0\omega_1 \times {}_0\mathbf{R}_1\ {}^1\vec{p}_Q) + {}^0\vec{v}_1 $$

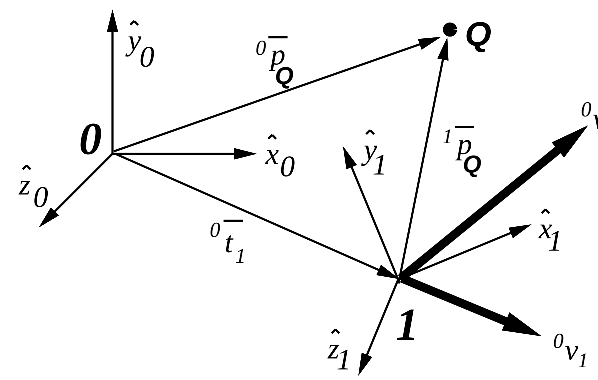


Figure 4.9 *Propagating velocity and acceleration into a non-inertial coordinate frame.*

The linear acceleration of particle Q is then,

$$
\begin{aligned}
{}^0\dot{\vec{v}}_Q &= \frac{d}{dt}\left[{}_0\mathbf{R}_1\ {}^1\dot{\vec{p}}_Q \right] + ({}^0\dot{\omega}_1 \times {}_0\mathbf{R}_1\ {}^1\vec{p}_Q) + ({}^0\omega_1 \times \frac{d}{dt}\left[{}_0\mathbf{R}_1\ {}^1\vec{p}_Q \right]) + {}^0\dot{\vec{v}}_1 \\
&= {}_0\mathbf{R}_1\ {}^1\ddot{\vec{p}}_Q + ({}^0\omega_1 \times {}_0\mathbf{R}_1\ {}^1\dot{\vec{p}}_Q) + ({}^0\dot{\omega}_1 \times {}_0\mathbf{R}_1\ {}^1\vec{p}_Q) + ({}^0\omega_1 \times {}_0\mathbf{R}_1\ {}^1\dot{\vec{p}}_Q) \\
&\quad +({}^0\omega_1 \times {}^0\omega_1 \times {}_0\mathbf{R}_1\ {}^1\vec{p}_Q) + {}^0\dot{\vec{v}}_1
\end{aligned}
$$

By associating the i^{th} coordinate frame in a kinematic chain with frame 1 in Figure 4.9 (and therefore frames 0 and 2 become frames $(i-1)$ and $(i+1)$, respectively) and collecting terms, we may write

$$
\begin{aligned}
{}^{i+1}\dot{\vec{v}}_{i+1} &= {}_{i+1}R_{i-1} \Big[{}_{i-1}R_i\ {}^i\ddot{\vec{p}}_{i+1} + 2({}^{i-1}\omega_i \times {}_{i-1}R_i\ {}^i\dot{\vec{p}}_{i+1}) + ({}^{i-1}\dot{\omega}_i \times {}_{i-1}R_i\ {}^i\vec{p}_{i+1}) \\
&\qquad +({}^{i-1}\omega_i \times {}^{i-1}\omega_i \times {}_{i-1}R_i\ {}^i\vec{p}_{i+1}) + {}^{i-1}\dot{\vec{v}}_i \Big] \\
&= {}_{i+1}R_i \Big[{}^i\ddot{\vec{p}}_{i+1} + 2({}^i\omega_i \times {}^i\dot{\vec{p}}_{i+1}) + ({}^i\dot{\omega}_i \times {}^i\vec{p}_{i+1}) + ({}^i\omega_i \times {}^i\omega_i \times {}^i\vec{p}_{i+1}) + {}^i\dot{\vec{v}}_i \Big] \qquad (4.7)
\end{aligned}
$$

Equation 4.7 can be simplified somewhat given the type (prismatic or revolute) of the $i+1^{st}$ degree of freedom.

$$
\begin{aligned}
REVOLUTE: \quad & {}^i\vec{p}_{i+1} = const,\ {}^i\dot{\vec{p}}_{i+1} = {}^i\ddot{\vec{p}}_{i+1} = 0 \\
& {}^{i+1}\dot{\vec{v}}_{i+1} = {}_{i+1}R_i \Big[{}^i\dot{\vec{v}}_i + ({}^i\dot{\omega}_i \times {}^i\vec{p}_{i+1}) + ({}^i\omega_i \times {}^i\omega_i \times {}^i\vec{p}_{i+1}) \Big]
\end{aligned}
$$

$$
\begin{aligned}
PRISMATIC: \quad & {}^i\vec{p}_{i+1} = d_i\hat{x}_i,\ {}^i\dot{\vec{p}}_{i+1} = \dot{d}_i\hat{x}_i,\ {}^i\ddot{\vec{p}}_{i+1} = \ddot{d}_i\hat{x}_i \\
& {}^{i+1}\dot{\vec{v}}_{i+1} = {}_{i+1}R_i \Big[{}^i\dot{\vec{v}}_i + \ddot{d}_i\hat{x}_i + 2({}^i\omega_i \times \dot{d}_i\hat{x}_i) + ({}^i\dot{\omega}_i \times d_i\hat{x}_i) + ({}^i\omega_i \times {}^i\omega_i \times d_i\hat{x}_i) \Big]
\end{aligned}
$$

Having established $(\omega, \dot{\omega}, \dot{v})$ for link i, we must now refer the accelerations to the link's center of mass, where we can describe both the accelerations $(\dot{\omega}, \dot{v})$ and consequently, the net forces and moments (F, N) that are consistent with these accelerations.

$$ {}^{i+1}\dot{v}_{cm,(i+1)} = ({}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}\vec{p}_{cm}) + ({}^{i+1}\omega_{i+1} \times {}^{i+1}\omega_{i+1} \times {}^{i+1}\vec{p}_{cm}) + {}^{i+1}\dot{v}_{i+1} \quad (4.8) $$

$$ {}^{i+1}F_{i+1} = m_{i+1}\, {}^{i+1}\dot{v}_{cm\ i+1} \quad (4.9) $$

$$ {}^{i+1}N_{i+1} = M_{i+1}\, {}^{i+1}\dot{\omega}_{i+1} + ({}^{i+1}\omega_{i+1} \times M_{i+1}\, {}^{i+1}\omega_{i+1}) \quad (4.10) $$

4.5.2 Propagating Force in Open Kinematic Chains

Section 4.5.1 establishes that velocities are inherited from link to link starting from an inertial frame and culminating in the velocity at the end of the kinematic chain. The same is true of forces in the mechanism. If our mechanism is moving in freespace, then the distal unitary link, link n, in the chain sees only those forces accounted for in its state of motion and applied to its proximal end by link $n-1$.

As we move along the kinematic chain, distal to proximal, each successive link inherits a component of force applied to its distal end. Just as velocity propagates from inertial frame (where velocities are zero) outward, forces propagate from the free end of the manipulator (where forces are zero) inward. Figure 4.10 shows link i in a kinematic chain. We define ${}^{i}f_i$ and ${}^{i}\eta_i$ as the force and torque respectively, exerted on link i by link $i-1$. Furthermore, the superscript designates that these quantities are written in frame i coordinates. If we assume that the structure is in static equilibrium, then

$$ \sum Forces = {}^{i}F_i = {}^{i}f_i - {}_{i}R_{i+1}\, {}^{i+1}f_{i+1}, \text{ or} $$
$$ {}^{i}f_i = {}^{i}F_i + {}_{i}R_{i+1}\, {}^{i+1}f_{i+1} \quad (4.11) $$

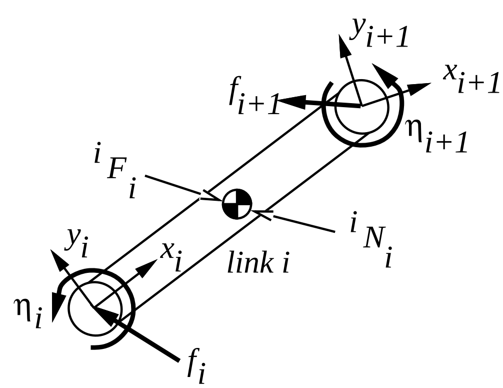


Figure 4.10 *The propagation of forces in a kinematic chain.*

$$ \sum Torques = {}^{i}N_i = {}^{i}\eta_i - {}^{i}\eta_{i+1} - ({}^{i}p_{cm} \times {}^{i}f_i) - (({}^{i}p_{i+1} - {}^{i}p_{cm}) \times {}^{i}f_{i+1}, $$

but, ${}^{i}f_i = {}^{i}F_i + {}_{i}R_{i+1}\, {}^{i+1}f_{i+1}$, by Equation 4.11 so that,

$$ {}^{i}N_i = {}^{i}\eta_i - {}^{i}\eta_{i+1} - ({}^{i}p_{cm} \times {}^{i}F_i) - ({}^{i}p_{cm} \times {}^{i}f_{i+1}) - ({}^{i}p_{i+1} \times {}^{i}f_{i+1}) + ({}^{i}p_{cm} \times {}^{i}f_{i+1}) $$
$$ = {}^{i}\eta_i - {}^{i}\eta_{i+1} - ({}^{i}p_{cm} \times {}^{i}F_i) - ({}^{i}p_{i+1} \times {}^{i}f_{i+1}) $$

or,

$$ {}^{i}\eta_i = {}^{i}N_i + {}_{i}R_{i+1}\, {}^{i+1}\eta_{i+1} + ({}^{i}p_{cm} \times {}^{i}F_i) + ({}^{i}p_{i+1} \times {}_{i}R_{i+1}\, {}^{i+1}f_{i+1}) \quad (4.12) $$

To summarize, the Newton-Euler equations are solved iteratively starting from the inertial frame ($\dot{v} = \dot{\omega} = 0$) and proceeding from proximal to distal. This procedure propagates velocities and accelerations from link to link and solves for the net force (${}^{i}F_i$) and moment (${}^{i}N_i$) on the center of mass of each link in the kinematic chain. Having finished this *outward* iteration, we begin an *inward* iteration starting from the free (distal) end of the manipulator, where ${}^{i+1}f_{i+1} = {}^{i+1}\eta_{i+1} = 0$. Equations 4.11 and 4.12 are applied to resolve the force and torque on the proximal end of link i. The forces and moments at frame i result in mechanical strains in a link or a joint, or they produce translational or rotational accelerations in the system. In the case of revolute joint i with axis of rotation about the $\hat{z}_i$ axis, the component of the torque about the $\hat{z}_i$ axis

$$\tau_i = {}^{i}\eta_i \cdot \hat{z}_i \tag{4.13}$$

produces an angular acceleration of the joint, whereas other components of the torque at frame i are resisted in the mechanism and propagated into other portions of the structure.

The component relationships of the outward-inward iteration for a robot composed of revolute joints exclusively are summarized in Tables 4.1 and 4.2.

Table 4.1: Outward Iteration Equations

Angular Velocity: ω — Equation 4.5.1

REVOLUTE: $${}^{i+1}\omega_{i+1} = {}_{i+1}\mathbf{R}_i\, {}^{i}\omega_i + \dot{\theta}_{i+1}\hat{z}_{i+1}$$

PRISMATIC: $${}^{i+1}\omega_{i+1} = {}_{i+1}\mathbf{R}_i\, {}^{i}\omega_i$$

Angular Acceleration: $\dot{\omega}$ — Equation 4.5.1

REVOLUTE: $${}_{i+1}\mathbf{R}_i\, {}^{i}\dot{\omega}_i + ({}_{i+1}\mathbf{R}_i\, {}^{i}\omega_i \times \dot{\theta}_{i+1}\hat{z}_{i+1}) + \ddot{\theta}_{i+1}\hat{z}_{i+1}$$

PRISMATIC: $${}^{i+1}\dot{\omega}_{i+1} = {}_{i+1}\mathbf{R}_i\, {}^{i}\dot{\omega}_i$$

Linear Acceleration: $\dot{v}$ — Equation 4.5.1

REVOLUTE: $${}^{i+1}\dot{\vec{v}}_{i+1} = {}_{i+1}R_i \left[{}^{i}\dot{\vec{v}}_i + ({}^{i}\dot{\omega}_i \times {}^{i}\vec{p}_{i+1}) + ({}^{i}\omega_i \times {}^{i}\omega_i \times {}^{i}\vec{p}_{i+1})\right]$$

PRISMATIC: $${}^{i+1}\dot{\vec{v}}_{i+1} = {}_{i+1}R_i \Big[{}^{i}\dot{\vec{v}}_i + \ddot{d}_i\hat{x}_i + 2({}^{i}\omega_i \times \dot{d}_i\hat{x}_i) + ({}^{i}\dot{\omega}_i \times d_i\hat{x}_i) + ({}^{i}\omega_i \times {}^{i}\omega_i \times d_i\hat{x}_i)\Big]$$

Linear Acceleration (center of mass): $\dot{v}_{cm}$ — Equation 4.8

$${}^{i+1}\dot{v}_{cm,(i+1)} = ({}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}\vec{p}_{cm}) + ({}^{i+1}\omega_{i+1} \times {}^{i+1}\omega_{i+1} \times {}^{i+1}\vec{p}_{cm}) + {}^{i+1}\dot{v}_{i+1}$$

Net Force: F — $${}^{i+1}F_{i+1} = m_{i+1}\, {}^{i+1}\dot{v}_{cm\ i+1}$$ — Equation 4.9

Net Moment: N — $${}^{i+1}N_{i+1} = M_{i+1}\, {}^{i+1}\dot{\omega}_{i+1} + ({}^{i+1}\omega_{i+1} \times M_{i+1}\, {}^{i+1}\omega_{i+1})$$ — Equation 4.10

Table 4.2: Inward Iteration Equations

Inter-Link Forces:	${}^{i}f_i = {}^{i}F_i + {}_{i}R_{i+1}\,{}^{i+1}f_{i+1}$	Equation 4.11
Inter-Link Moments:	${}^{i}\eta_i = {}^{i}N_i + {}_{i}R_{i+1}\,{}^{i+1}\eta_{i+1} + ({}^{i}p_{cm} \times {}^{i}F_i) + ({}^{i}p_{i+1} \times {}_{i}R_{i+1}\,{}^{i+1}f_{i+1})$	Equation 4.12

4.6 Lagrangian Mechanics — Equations of Motion

The Lagrangian L is defined to be the difference between the kinetic energy T and the potential energy V of a dynamical system.

$$L(q,\dot{q}) = T(q,\dot{q}) - V(q)$$

This simple expression, together with results from the calculus of variations, yields an elegant means for constructing the dynamic equations of motion. To compute Langrange's equations, we will make use the following.

Theorem: (Lagrange's Equations) The equations of motion for a mechanical syste with generalized coordinates $q \in \mathcal{R}^m$ and Lagrangian, L are given by:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \Upsilon_i \qquad i = 1, \ldots, m$$

where Υ_i is the vector of external forces acting on the i^{th} generalized coordinate, g_i.

In vector coordinates,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{q}}} - \frac{\partial L}{\partial \vec{q}} = \vec{\Upsilon}$$

The proof of Equation 4.6 involves the Calculus of Variations and is beyond the scope of this text.

The dynamic equations of motion are obtained from:

$$\Upsilon_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$$

where the q_i are the configuration variables, $\dot{q}_i$ are the corresponding velocities, and F_i are the related forces or torques.

To gain some physical intuition into the Lagrange's equation, suppose that kinetic energy terms look like $\frac{1}{2}m\dot{q}^2$ and that potential energies are considered of the form mgq. Then, inserting these kinds of energy terms into the Lagrangian, we find:

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] &= \frac{\partial L}{\partial q} + \Upsilon \\ \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}m\dot{q}^2\right)\right] &= \frac{\partial}{\partial q}(mgq) + \Upsilon \\ \frac{d}{dt}(m\dot{q}) &= mg + \Upsilon \end{aligned}$$

Or in words, the net external force on the system (right-hand side) is equivalent to the time rate of change in the momentum (left-hand side) of the system. This is just Newton's equation, so we may be assured that Newtonian mechanics and Lagrangian mechanics produce the same kind of results.

4.7 Structure of the Dynamic Equations

In general, the equations of motion for an n degree of freedom open chain mechanisms can be written in the following form:

$$\tau = M(q)\ddot{q} + V(q,\dot{q}) + G(q),$$

where $M(q)$ is the $n \times n$ (symmetric and positive definite[2]) configuration dependent inertia matrix that relates acceleration and torque, $V(q, \dot{q})$ is the $n \times 1$ vector of velocity dependent torques (centrifugal and Coriolis terms), and G is a $n \times 1$ matrix containing all gravitational forces. Equation 4.7 is refered to as the **state-space form** since it is organized by the state variables $(q, \dot{q})$.

If we consider the planar, 2R robot shown in Figure 4.11, then

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

where, $\Theta = (\theta 1, \theta 2)$, M is a 2×2 inertia matrix, V is a 2×1 vector or torques representing the velocity terms, and and G is a 2×1 vector of gravitational torques. The Newton-Euler equations produce exactly the same equations of motion as the Langrangian formulation for this system. The results for either analysis yield:

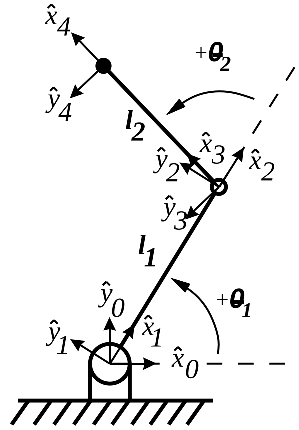


Figure 4.11 *The Two Degree of Freedom, Planar Robot.*

$$M(\Theta) = \begin{bmatrix} l_2^2 m_2 + 2 l_1 l_2 m_2 c_2 + l_1^2 (m_1 + m_2) & l_2^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2^2 m_2 + l_1 l_2 m_2 c_2 & l_2^2 m_2 \end{bmatrix},$$

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix},$$

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}.$$

To compute the configuration of a system described by Equation 4.7, we will re-write it in the form:

$$\ddot{\Theta} = M^{-1}(\Theta)\left[\tau - V(\Theta, \dot{\Theta}) - G(\Theta) - F\right]$$

where F is a catch-all term for representing any miscellaneous external force (contact loads, friction, etc). Now, the state of motion together with command torques is used to compute the resulting acceleration in the robot mechanism. Real robots are physical devices that perform this computation in analog, but Equation 4.7 can be used to simulate the mechanism. Given suitable initial state variables, the state of the system can be predicted for all future time. For example, using a simple Euler integrator, we find:

$$\begin{aligned} \ddot{\Theta}(t) &= M^{-1}[\tau - V - G - F] \\ \dot{\Theta}(t + \Delta t) &= \dot{\Theta}(t) + \ddot{\Theta}(t)\Delta t \\ \Theta(t + \Delta t) &= \Theta(t) + \dot{\Theta}\Delta t + \frac{1}{2}\ddot{\Theta}(t)\Delta t^2 \end{aligned}$$

[2]and is therefore always invertible

This is in fact, how dynamic simulations work, although better integrators are usually employed.

A final observation is warranted before we move on concerning the use of our dynamic model in the control of the system. Consider the compensated system illustrated in Figure 4.12.

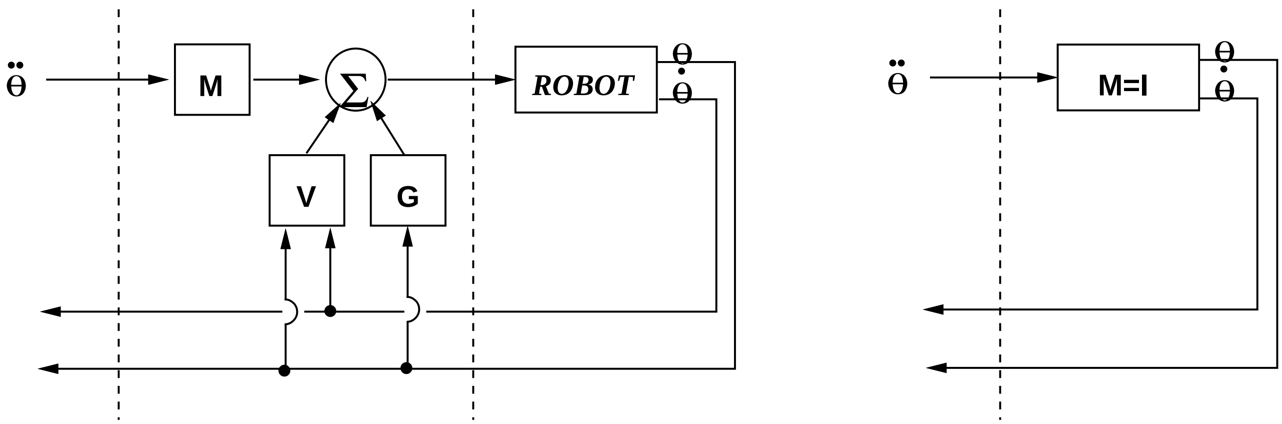
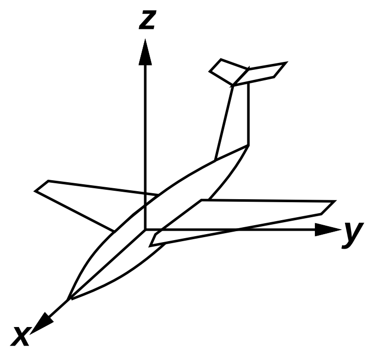


Figure 4.12 *Compensating to Linearize and Decouple Complex, Nonlinear Plants.*

The incorporation of the feedforward compensator produces a dynamical system that approximates the identity plant. This means that every mass in the system behaves as if it is a simple unit inertia, and that its pattern of motion is independent of the motion in other systems masses. If we have succeeded, that is, if the model on which the compensator is based is a good accounting of the real system, then from the perspective of control, we are left with the system shown on the right side of Figure 4.12. We will see in subsequent chapters, that we may exploit this result when we compute $\ddot{\theta}(t)$ trajectories for the system.

4.8 Homework Exercises

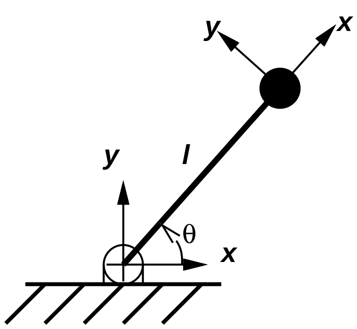
1. The inertia matrix of an airplane with respect to the xyz coordinate system at its mass center as shown in the following figure.



$$I = \begin{bmatrix} 100,000 & 0 & 20,000 \\ 0 & 150,000 & 0 \\ 20,000 & 0 & 250,000 \end{bmatrix}$$

Locate the principal moments of inertia Note that the x axis is longitudinal and the y axis is lateral.

2. Derive the dynamic equation of motion for the 1 DOF system illustrated.



Write the dynamic equation of motion in the state space form for this system, using:

(a) Newton/Euler equations (outward/inward iterations).

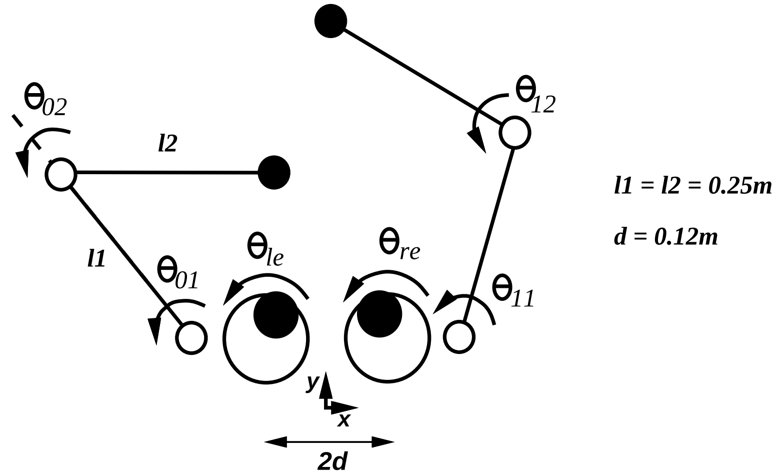
(b) Lagrangian formulation.

3. Derive the dynamic equation of motion for the cylindrical robot described in problem 3.6 using the Lagrangian formulation.

4. This problem concerns the Roger-the-Crab mechanism[3] and simulator that we will use a various points throughout the course.

[3]This creature was originally described in:

Churchland, P.M., *Matter and Consciousness: A Contemporary Introduction to the Philosophy of Mind*, Bradford/MIT Press, Cambridge, MA, 1988.



Roger lives in a "flat-land" world where gravitational forces act in the negative y direction. Roger's eyes and arm are dynamic systems and this homework problem involves modeling the inertial parameters of Roger's world. This is analogous to "uncrating" a robot and identifying the influence of gravity and inertia on the motion of the system before you actually do anything with it.

Problem 2 concerned the dynamic equation of motion for the 1 DOF eye mechanism. The dynamics of the arm are a bit more complicated:

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

where, $\Theta = (\theta 1, \theta 2)$, M is a 2×2 (symmetric and positive definite[4]) inertia matrix,

$$M(\Theta) = \begin{bmatrix} l_2^2 m_2 + 2 l_1 l_2 m_2 c_2 + l_1^2 (m_1 + m_2) & l_2^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2^2 m_2 + l_1 l_2 m_2 c_2 & l_2^2 m_2 \end{bmatrix},$$

V is a 2×1 vector incorporating all terms which depend on velocity in the system (centrifugal and Coriolis forces),

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix},$$

and G is a 2×1 matrix containing all gravitational forces,

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix},$$

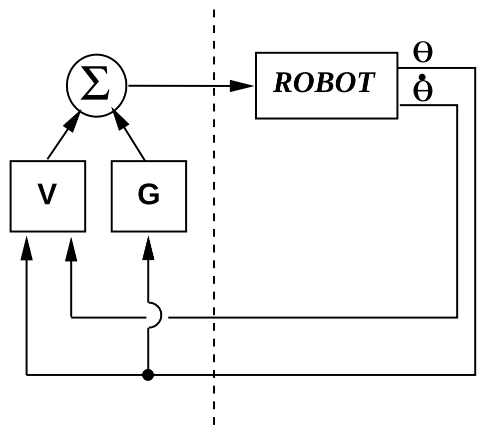
[4]The symmetric and positive definite properties provide that the M matrix is always invertible

(a) **System Identification**

Design an experimental procedure for identifying the parameters of the dynamic equation of motion of the eyes and the arm. Describe the experimental procedure in detail and the results you obtain.

(b) Copy the simulator in */courses/cs600/cs603/cs603/xroger* to your directory. This code consists of a README file that describes the simulation briefly, and control files *control.c* and *control.h*. For this (and subsequent) homework problems using Roger, you will have to add code to the *control.c* file - no other file should be changed.

With the dynamics determined in problem 2 and given above, it is possible to build the so-called feed-forward compensator depicted here.



Implement these feedforward compensators for both the arm and the eyes in the simulator. Demonstrate that the result is linearized and decoupled analytically, and show evidense that the feed-forward compensator accomplishes this in the simulator.